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The third equilibrium equation for forces of flexural member cross-section

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1. Introduction

Values of stress-strain state parameters in (perpendicular to longitudinal axes) cross-sections for various levels of loading are required to determine in investigation and design of building structures [1-15]. For simplification of calculations the real stress diagrams are superseded by arbitrary ones [5-10] and therefore calculated values of parameters are arbitrary as well. More realistic values are obtained using correction factors [7-9]. The factors are determined by experimental investigations. Different formulas are used for various loading stages [6-10].

When a member is subjected to the action of bending moments M and/or axial forces N (hereafter- flexural member) is symmetrical in relation to the plane of the longitudinal force and bending moments and axial forces act in the same plane and all forces are parallel then two equations of static equilibrium, for projections of forces and for moments, are available. In articles [11-14] a practical engineering and fairly general method (for simplicity hereafter referred to as ZI method) is presented which being based on nonlinear model enables using unified method calculation of real rather than arbitrary values of stress-strain state parameters for various cross-sections of variously reinforced members made of various materials at any loading stage from the beginning of loading up to member failure. It is convenient to use two involved here equations of static (for projections of forces and for moments) equilibrium in the case when strain of any layer of the member is known in advance. In the case of cracking moment calculation strain of the layer subjected to the greatest tension is known. In the case of the ultimate moment of reinforced concrete member calculation strain of the layer subjected to the greatest compression (or that of the tensile reinforcement) is known. In such cases the main problem (and often the most complicated one) – determination of the neutral axis location – is readily solved. In verification problems location of the neutral axis is determined from the equation of projections of forces and in the design problems – from the equation of moments. There are many cases when external forces are given but strain of none layer is known. Such cases are encountered when stress-strain states of reinforced concrete members without cracks or these with cracks are examined. Then for example with reinforced concrete members one has to deal with three unknowns: a parameter (neutral axis depth x or its relative depth ξ), tensile reinforcement strain ε_s (or stress σ_s) and concrete compression strain ε_c (or stress σ_c). But there are only two equations of static equilibrium for actions. In STR [9] location of the neutral axis depth (value of ξ) is calculated by a formula obtained by tests. Obviously it suits for the cases of performed experiments from which the formula was developed. It is the shortcoming of an empirical formula. Moreover the formula is fairly complicated. Greatly simplified arbitrary rectangular stress distribution diagram in compression zone of the member was used for development of this formula. Concrete stress in the tension zone is neglected. Thus arbitrary rather than real values of ξ are obtained. In EN-2 method Bernoulli hypothesis of plane sections is used as the third equation. Additionally in EN-2 method simplified design diagrams are employed. Regulation EN-2 formula for curvilinear diagram of stresses in concrete compression zone is presented but the method of its use is not given. In the ZI method not only more realistic stress diagrams can be applied but possible deviations from the hypothesis of plane sections may be evaluated as well. In earlier published articles calculation by ZI method involves application of consecutive approach procedure when longitudinal forces $N \neq 0$ or/and attributed to them prestressing forces $P \neq 0$ requiring execution of consecutive approach cycles for calculation of two parameters, i. e. location of the neutral axis and strain value of a chosen layer [11, 16]. It is essential to simplification of calculations availability of a third theoretic equation for determination of real ξ value when strain value of any member layer is not known in advance and $N \neq 0$ or/and $P \neq 0$ as well.

Object of investigation – flexural members with symmetry plane, passing longitudinal axis, in which bending moments and/or axial forces act (Fig. 1).

The main goal of investigation – proposition for ZI method the third general mathematic equation for static equilibrium of forces and bending moments acting in the cross-section making it possible to calculate according to a unified method location of the neutral axis for a flexural member of any cross-section, made of any material, with various reinforcement and for any stage of loading – from beginning of loading up to member failure. For simplicity this equation further will be referred to as the third static equilibrium equation of actions (hereafter – usually as the third equation). Thus, the task of this work is to derive a theoretic solution for very important problem and in many cases the most complicated one – calculation of neutral axes location by ZI method for cross-section of flexural members when strain of any layer is not known beforehand together with $N \neq 0$ or/and $P \neq 0$.

Supplementary goal – to show haw the three fundamental equations for static equilibrium of ZI method can be applied for solution of practical problems and development of equation versions for the said purpose.

Tasks. Presentation versions of three equations for static equilibrium of forces as follows: 1) the most general version; 2) versions for I-section of reinforced concrete member with double reinforcement; 3) versions for rectangular cross-section member. Calculation of reinforced concrete members probably is more complicated than that of the members from other materials since reinforced concrete is a complex elastic plastic material. Members can be either without cracks or with cracks. Calculation can be performed for a section at crack or for that between the cracks.

When special values of coefficients are used in ZI method equations they become suitable for elastic plastic and for elastic material as well: concrete, plastics, timber, steel etc.

2. Essence of the method and formulae

Eq. (1) of projections of forces and Eq. (2) of bending moments in relation to an optional line $a-a$ (Fig. 1) are taken from the article [11]. In the said equations appears strain ε_ε of any optional layer at the distance a_ε from the axes $w-w$. When in Eqs. (1) and (2) $a_\varepsilon=0$ ($\varepsilon_\varepsilon/k_\varepsilon=\varepsilon_w$) and $a_a=0$ then the bending moments in Eq. (2) are taken about the axes $w-w$ ($a-a$ coincide with $w-w$ axes) and thus Eqs. (3) and (4) are obtained. When parameter $E\varepsilon_w$ is removed from Eqs. (3) and (4) Eq. (7) is obtained. It is common equation for projections of forces and for bending moments. It will be referred to as the third equation for static equilibrium of actions or in brief way – simply the third equation. Further it will be shown that application of these three equations (force projections, bending moments and the third equation) makes it possible solution of many various practical problems (see section 4).

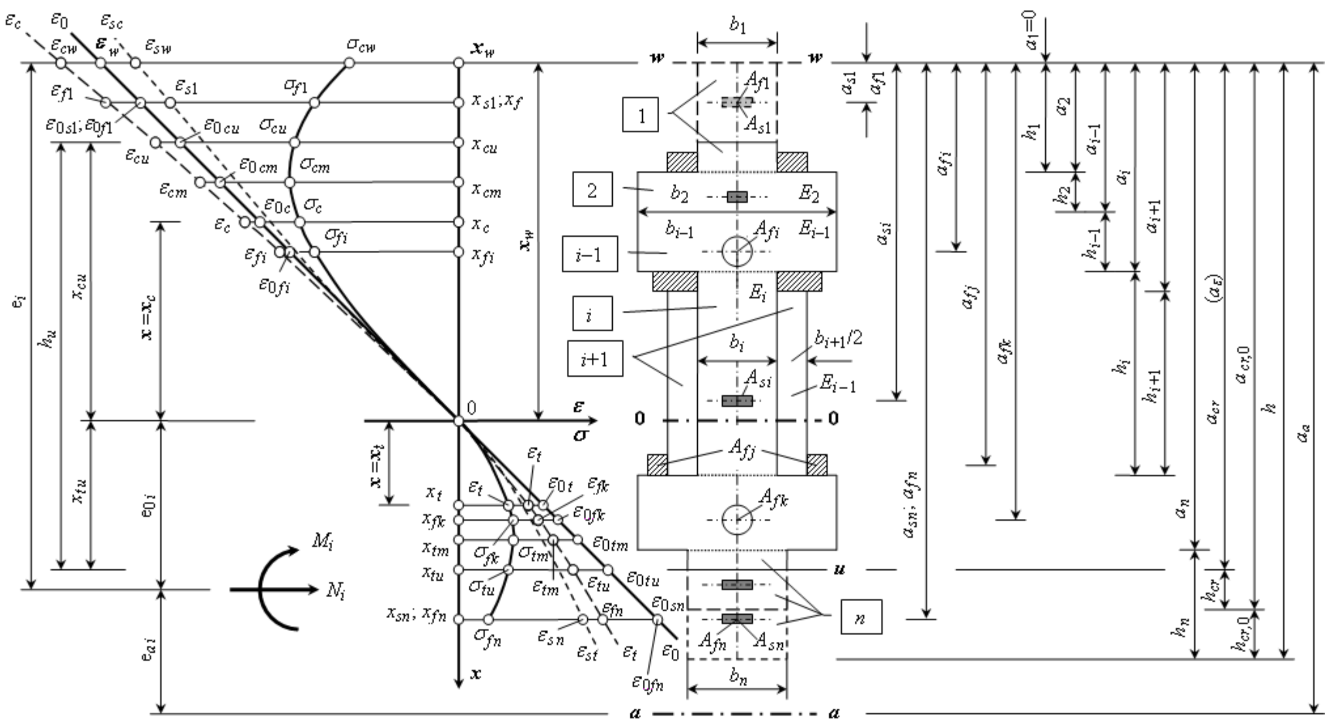

Fig. 1 Cross-section of the member and stress-strain diagrams

$$\sum k_i\alpha_{ei}b_i\left(\omega_{i2}-\omega_{i1}\right)x_w^2+\left[2\sum k_i\alpha_{ei}b_i\left(\omega_{i2}d_{iu}-\omega_{i1}a_{iu}\right)+\sum k_{fi}\alpha_{efi}A_{fi}\nu_{fi}+\sum k_{si}\alpha_{esi}A_{si}\nu_{Si}+\frac{\sum\left(P_i\nu_{Si}/\nu_{pi}\right)+\sum N_i}{E\varepsilon_\varepsilon/k_\varepsilon}\right]x_w+$$

$$+\sum k_i\alpha_{ei}b_i\left(\omega_{i2}d_{iu}^2-\omega_{i1}a_{iu}^2\right)+\sum k_{fi}\alpha_{efi}A_{fi}\nu_{fi}a_{fiu}+\sum k_{si}\alpha_{esi}A_{si}\nu_{Si}a_{siu}+\frac{\sum\left(P_i\nu_{Si}/\nu_{pi}\right)+\sum N_i}{E\varepsilon_\varepsilon/k_\varepsilon}a_\varepsilon=0; \qquad (1)$$

$$\sum k_i\alpha_{ei}b_i\left[\left(\omega_{i2}-\omega_{i1}\right)-\left(\varpi_{i2}-\varpi_{i1}\right)\right]x_w^3+\sum k_i\alpha_{ei}b_i\left[\left(\omega_{i2}-\omega_{i1}\right)a_a+2\left(\omega_{i2}d_{iu}-\omega_{i1}a_{iu}\right)-3\left(\varpi_{i2}d_{iu}-\varpi_{i1}a_{iu}\right)\right]x_w^2+$$

$$+\left\{\begin{array}{l}\sum k_i\alpha_{ei}b_i\left[2\left(\omega_{i2}d_{iu}-\omega_{i1}a_{iu}\right)a_a+\left(\omega_{i2}d_{iu}^2-\omega_{i1}a_{iu}^2\right)-3\left(\varpi_{i2}d_{iu}^2-\varpi_{i1}a_{iu}^2\right)\right]+\\+\sum k_{fi}\alpha_{efi}A_{fi}\nu_{fi}\left(a_a-a_{fiu}\right)+\sum k_{si}\alpha_{esi}A_{si}\nu_{Si}\left(a_a-a_{siu}\right)+\dfrac{\sum\left(P_i\nu_{Si}/\nu_{pi}\right)\left(a_a-a_{siu}\right)+\sum N_i\left(a_a-e_i\right)+\sum M_i}{E\varepsilon_\varepsilon/k_\varepsilon}\end{array}\right\}x_w+$$

$$+\sum k_i\alpha_{ei}b_i\left[\left(\omega_{i2}d_{iu}^2-\omega_{i1}a_{iu}^2\right)a_a-\left(\varpi_{i2}d_{iu}^3-\varpi_{i1}a_{iu}^3\right)\right]+\sum k_{fi}\alpha_{efi}A_{fi}\nu_{fi}a_{fiu}\left(a_a-a_{fiu}\right)+$$

$$+\sum k_{si}\alpha_{esi}A_{si}\nu_{Si}a_{siu}\left(a_a-a_{siu}\right)+\frac{\sum\left(P_i\nu_{Si}/\nu_{pi}\right)\left(a_a-a_{siu}\right)+\sum N_i\left(a_a-e_i\right)+\sum M_i}{E\varepsilon_\varepsilon/k_\varepsilon}a_\varepsilon=0; \qquad (2)$$

$$\sum k_i\alpha_{ei}b_i\left(\omega_{i2}-\omega_{i1}\right)x_w^2+\left[2\sum k_i\alpha_{ei}b_i\left(\omega_{i2}d_{iu}-\omega_{i1}a_{iu}\right)+\sum k_{fi}\alpha_{efi}A_{fi}\nu_{fi}+\sum k_{si}\alpha_{esi}A_{si}\nu_{Si}+\frac{\sum\left(P_i\nu_{Si}/\nu_{pi}\right)+\sum N_i}{E\varepsilon_w}\right]x_w+$$
$$+\sum k_i\alpha_{ei}b_i\left(\omega_{i2}d_{iu}^2-\omega_{i1}a_{iu}^2\right)+\sum k_{fi}\alpha_{efi}A_{fi}\nu_{fi}a_{fiu}+\sum k_{si}\alpha_{esi}A_{si}\nu_{Si}a_{siu}=0\,; \tag{3}$$

$$\sum k_i\alpha_{ei}b_i\left[\left(\varpi_{i2}-\varpi_{i1}\right)-\left(\omega_{i2}-\omega_{i1}\right)\right]x_w^3+\sum k_i\alpha_{ei}b_i\left[3\left(\varpi_{i2}d_{iu}-\varpi_{i1}a_{iu}\right)-2\left(\omega_{i2}d_{iu}-\omega_{i1}a_{iu}\right)\right]x_w^2+$$
$$+\left\{\begin{array}{l}\sum k_i\alpha_{ei}b_i\left[3\left(\varpi_{i2}d_{iu}^2-\varpi_{i1}a_{iu}^2\right)-\left(\omega_{i2}d_{iu}^2-\omega_{i1}a_{iu}^2\right)\right]+\sum k_{fi}\alpha_{efi}A_{fi}\nu_{fi}a_{fiu}\\+\sum k_{si}\alpha_{esi}A_{si}\nu_{Si}a_{siu}+\dfrac{\sum\left(P_i\nu_{Si}/\nu_{pi}\right)a_{siu}+\sum N_ie_i-\sum M_i}{E\varepsilon_w}\end{array}\right\}x_w+$$
$$+\sum k_i\alpha_{ei}b_i\left(\varpi_{i2}d_{iu}^3-\varpi_{i1}a_{iu}^3\right)+\sum k_{fi}\alpha_{efi}A_{fi}\nu_{fi}a_{fiu}^2+\sum k_{si}\alpha_{esi}A_{si}\nu_{Si}a_{siu}^2=0\,. \tag{4}$$

This article is continuation of the article [11]. Notations of parameters in Eqs. (1)-(4) are the same as in the article [11]: $\overline{E}_i=\nu_iE_i=\nu_i\alpha_{ei}E$; $\overline{E}_{fi}=\nu_{fi}E_{fi}=\nu_{fi}\alpha_{efi}E$; $\overline{E}_{si}=\nu_{Si}E_{si}=\nu_{Si}\alpha_{esi}E$; $\alpha_{ei}=E_i/E$; $\alpha_{efi}=E_{fi}/E$; $\alpha_{esi}=E_{si}/E$; $\varepsilon_{Si}=\varepsilon_{pi}+\varepsilon_{si}$, $\sigma_{Si}=\sigma_{pi}+\sigma_{si}$; $\nu_{Si}=\sigma_{Si}/\overline{E}_{si}\varepsilon_{Si}$; $\nu_{pi}=\sigma_{pi}/\overline{E}_{si}\varepsilon_{pi}$; $k_i=\varepsilon_i/\varepsilon_{0i}$; $k_{fi}=\varepsilon_{fi}/\varepsilon_{0fi}$; $k_{si}=\varepsilon_{si}/\varepsilon_{0si}$; $k_\varepsilon=\varepsilon_\varepsilon/\varepsilon_{0\varepsilon}$; $\varphi=\dfrac{\varepsilon_w}{x_w}=\dfrac{\varepsilon_{0i}}{x_i}=\dfrac{\varepsilon_{0fi}}{x_{fi}}=\dfrac{\varepsilon_{0si}}{x_{si}}=\dfrac{\varepsilon_{0mi}}{x_{mi}}$; $d_{iu}=a_i+h_i$ (Fig. 1). $P_i=\sigma_{pi}A_{si}=\overline{E}_{si}A_{si}\varepsilon_{pi}=\nu_{pi}E_{si}A_{si}\varepsilon_{pi}$. Letter i in subscripts of parameters indicates number of the element, letter u shows that value of parameters not greater and not less than its limit value is used [11]. Z_c and Z_t – compression and tension zones in the effective cross-section of the member; for the zone Z_c compression material characteristics are used and for the zone Z_t – tension material characteristics. For Z_c zone values of strains, strengths of materials and x_w are negative (see Fig. 1 and [11]). Number 1 in parameter subscripts indicate edge of a rectangular member subjected to either the greatest compression or the least tension, i.e. the edge where mathematically of minimal value stresses σ_i, while number 2 – denotes edge of rectangular member subjected either to the least compression or to the greatest tension, i. e. the edge at which mathematically the greatest σ_i, act. a_i and $d_i=a_i+h_i$ are distances from axis $w-w$ to the rectangular member edge denoted by number i, i.e. corresponding distances to the closer edge 1 and farther edge 2 to the axis $w-w$. h_i is the depth of the member *i*. Common subscript for edges 1 and 2 is *j*.

In calculations of $\eta_i=\eta_{i1}$, $\omega_i=\omega_{i1}$ and $\varpi_i=\varpi_{i1}$, $\overline{x}_i=\overline{x}_{i1}$ is taken; for calculation of $\eta_i=\eta_{i2}$, $\omega_i=\omega_{i2}$ and $\varpi_i=\varpi_{i2}$, it is taken $\overline{x}_i=\overline{x}_{i2}$. $x_{i1}=a_i+x_w$, $\overline{x}_{i1}=x_{i1}$, but $x_{cu}\le\overline{x}_{i1}\le x_{tu}$. $d_i=a_i+h_i$ (see Fig. 1); $x_{i2}=d_i+x_w$, $\overline{x}_{i2}=x_{i2}$, but $x_{cu}\le\overline{x}_{i2}\le x_{tu}$. $x_{cu}=\dfrac{\varepsilon_{0cu}}{\varphi}$; $x_{tu}=\dfrac{\varepsilon_{0tu}}{\varphi}$, but $x_{tu}\le a_{cr}-x_w$; here: $a_{cr}=a_{cr,0}-h_{cr}=h-h_{cr,0}-h_{cr}$; $a_{min}=x_{cu}-x_w$, $a_{max}=x_{tu}-x_w$; $a_{iu}=a_i$, but $a_{min}\le a_{iu}\le a_{max}$; $d_{iu}=d_i$, but $a_{min}\le d_{iu}\le a_{max}$. $\overline{x}_{i1}=a_{iu}+x_w$. $\overline{x}_{i2}=d_{iu}+x_w$.

Using notations:

$$\kappa_{n2}=\sum k_i\alpha_{ei}b_i\left(\omega_{i2}-\omega_{i1}\right);$$
$$\kappa_{n1}=2\sum k_i\alpha_{ei}b_i\left(\omega_{i2}d_{iu}-\omega_{i1}a_{iu}\right)+$$
$$+\sum k_{fi}\alpha_{efi}A_{fi}\nu_{fi}+\sum k_{si}\alpha_{esi}A_{si}\nu_{Si};$$
$$\kappa_{n0}=\sum k_i\alpha_{ei}b_i\left(\omega_{i2}d_{iu}^2-\omega_{i1}a_{iu}^2\right)+$$
$$+\sum k_{fi}\alpha_{efi}A_{fi}\nu_{fi}a_{fiu}+\sum k_{si}\alpha_{esi}A_{si}\nu_{Si}a_{siu};$$
$$\kappa_n=\sum\left(P_i\nu_{Si}/\nu_{pi}\right)+\sum N_i;$$
$$\kappa_{m3}=\sum k_i\alpha_{ei}b_i\left[\left(\varpi_{i2}-\varpi_{i1}\right)-\left(\omega_{i2}-\omega_{i1}\right)\right];$$
$$\kappa_{m2}=\sum k_i\alpha_{ei}b_i\left[3\left(\varpi_{i2}d_{iu}-\varpi_{i1}a_{iu}\right)-2\left(\omega_{i2}d_{iu}-\omega_{i1}a_{iu}\right)\right];$$
$$\kappa_{m1}=\sum k_i\alpha_{ei}b_i\left[3\left(\varpi_{i2}d_{iu}^2-\varpi_{i1}a_{iu}^2\right)-\left(\omega_{i2}d_{iu}^2-\omega_{i1}a_{iu}^2\right)\right]+$$
$$+\sum k_{fi}\alpha_{efi}A_{fi}\nu_{fi}a_{fiu}+\sum k_{si}\alpha_{esi}A_{si}\nu_{Si}a_{siu};$$
$$\kappa_{m0}=\sum k_i\alpha_{ei}b_i\left(\varpi_{i2}d_{iu}^3-\varpi_{i1}a_{iu}^3\right)+$$
$$+\sum k_{fi}\alpha_{efi}A_{fi}\nu_{fi}a_{fiu}^2+\sum k_{si}\alpha_{esi}A_{si}\nu_{Si}a_{siu}^2;$$
$$\kappa_m=\sum\left(P_i\nu_{Si}/\nu_{pi}\right)a_{siu}+\sum N_ie_i-\sum M_i,$$

Eqs. (3) and (4) attain the shape:

$$\kappa_{n2}x_w^2+\left(\kappa_{n1}+\frac{\kappa_n}{E\varepsilon_w}\right)x_w+\kappa_{n0}=0\,; \tag{5}$$

$$\kappa_{m3}x_w^3+\kappa_{m2}x_w^2+\left(\kappa_{m1}+\frac{\kappa_m}{E\varepsilon_w}\right)x_w+\kappa_{m0}=0\,. \tag{6}$$

Exclusion of $E\varepsilon_w$ from Eqs. (5) and (6) the most general Eq. (7) for calculation of the neutral axis location x_w or ξ_w in any stress-strain state (for any loading stage) in the cases when external actions *M*, *N* and prestressing force *P* are given but of ε_c (and σ_c) also ε_s (and σ_s) are not given, parameter values:

$$\kappa_{m3}\kappa_nx_w^3+\left(\kappa_{m2}\kappa_n-\kappa_{n2}\kappa_m\right)x_w^2+$$
$$+\left(\kappa_{m1}\kappa_n-\kappa_{n1}\kappa_m\right)x_w+\left(\kappa_{m0}\kappa_n-\kappa_{n0}\kappa_m\right)=0\,. \tag{7}$$

For Eqs. (5)-(7) $a_\varepsilon=0$ and $a_a=0$ – the bending moments in respect to $w-w$ are taken.

Eq. (7) is the third equation of static equilibrium of forces. Together with two the first ones (1) and (2) or (3) and (4), or (5) and (6) equations make a system of three equations and enable to solve many problems in analytical way (see section 4). When $\kappa_n = 0$ then location of the neutral axis (x_w) can be calculated using either one of Eqs. (1), (3) and (5) or from Eq. (7). When $\kappa_n \neq 0$ then x_w is calculated from Eq. (7).

Stresses of the main material of the member are readily described by a simply-to-integrate polynomial (higher degree multinomial) [11]:

$$\sigma_i = E_i\varepsilon_i\nu_i = E_i\varepsilon_i\left(1 + c_{1i}\eta_i + c_{2i}\eta_i^2 + c_{3i}\eta_i^3 + \cdots\right), \quad (8)$$

where $\eta_i = \varepsilon_i / \varepsilon_{mi}$ (Figs. 1 and 3), values of c_i see below. In formulas (9) and (10) $\varepsilon_i = \varepsilon_{ij}$, $\eta_i = \eta_{ij} = \varepsilon_{ij}/\varepsilon_{mi}$.

$$\omega_{ij} = \frac{1}{2} + \frac{c_{1i}}{3}\eta_{ij} + \frac{c_{2i}}{4}\eta_{ij}^2 + \frac{c_{3i}}{5}\eta_{ij}^3 + \frac{c_{4i}}{6}\eta_{ij}^4 + \cdots; \quad (9)$$

$$\varpi_{ij} = \frac{1}{3} + \frac{c_{1i}}{4}\eta_{ij} + \frac{c_{2i}}{5}\eta_{ij}^2 + \frac{c_{3i}}{6}\eta_{ij}^3 + \frac{c_{4i}}{7}\eta_{ij}^4 + \cdots. \quad (10)$$

When exact description of $\sigma_c - \varepsilon_c$ relationship for concrete (Fig. 2) is required not only of "ascending" its part but and of "descending" one as well then coefficient c_i values can be taken from articles [11, 17] and when only "ascending" part is required c_i values can by taken from [18]. In the first case function $\sigma_c - \varepsilon_c$ is of the 5th degree and in the second case it is much simpler – 3rd degree, namely:

$$\sigma_c = E_c\varepsilon_c\left(1 + c_1\eta + c_2\eta^2\right) = \nu_c E_c\varepsilon_c = \nu_c\sigma_{ce}; \quad (11)$$

$$\nu_c = 1 + c_1\eta + c_2\eta^2 = 1 + (3\nu_{c1} - 2)\eta + (1 - 2\nu_{c1})\eta^2, \quad (12)$$

$$E_c = tan\beta;\ \sigma_{ce1} = E_c\varepsilon_{c1};\ \sigma_{c1} = \sigma_{cm} = f_{cm};\ \nu_{c1} = \frac{\sigma_{c1}}{\sigma_{ce1}}.$$

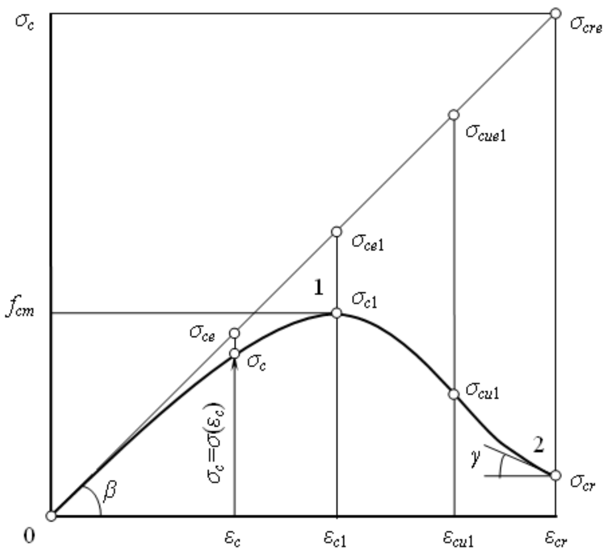

Fig. 2 Stress-strain relationship for concrete

In many cases 3rd degree function can be applied and for "descending" part as well, particularly for stronger concrete, but it needs verification.

Investigations performed by the author of this paper showed that for all common strength classes of concretes description by the 3rd degree function (11) of "ascending" part (up to ε_{c1}) of function in Fig. 2 is obtained quite exact when $\nu_{c1} \geq 0.3$ ($f_{ck} \geq 12$ MPa) and satisfactory – when $\nu_{c1} \geq 0.25$ ($f_{ck} \geq 08$ MPa).

Some part of the 3rd degree "descending" Eq. (11) may be applied when $\nu_{c1} \geq 0.39$, ($f_{ck} \geq 25$ MPa). The whole interval of "descending" part (up to ν_{cu1}) can be applied when $\nu_{c1} \geq 0.44$, ($f_{ck} \geq 35$ MPa).

3. Examples of practical application of ZI method possibilities and formulae

The most general and the most widely used in practice design cross-section shape of flexural members is I-section with reinforcement concentrated at sides of the member (Fig. 3).

Below equations are developed for the cases when:

1) hypothesis of plane sections (Bermoulli) for the whole cross-section; then $k_i = 1$;

2) the main material of the whole member is homogeneous and elasticity modulus is uniform; then $\alpha_{ei} = 1$;

3) cross-section is divided in rectangular parts in such a way that the depth of the web is equal to the total depth of the cross-section.

There are three rectangular parts of an I-section. Numbers and their respective subscripts of $b_c h_{fc} = (b_{fc} - b)h_{fc}$ – upper (in compression part) flange dimensions, bh – web dimensions and $b_t h_{ft} = (b_{ft} - b_b)h_{ft}$ – lower (in tension part) flange dimensions and further are indicated by respective letters *c*, *b* and *t*. In Fig. 1 and in the most general formulae of ZI method these subscripts are *i* = 1, *i* = 2 and *i* = 3 correspondingly.

Reinforcement is located at member edges only. Subscript of Z_t zone reinforcement is st and of Z_c zone – sc.

Since distances are taken in respect to $w-w$ axes (Fig. 1) then in Eqs. (1)–(7) values of $a_{iu} = a_{1u} = a_{cu} = 0$, $a_{iu} = a_{2u} = a_{bu} = 0$ and $a_{iu} = a_{3u} = a_{tu} = (h - h_{ft})_u$.

For this case the following symbols of values for parameters of general ZI method equations are applied:

$d_{iu} = d_{1u} = (a_1 + h_1)_u = d_{cu} = (a_c + h_{fc})_u = h_{fcu}$,

$d_{iu} = d_{2u} = (a_2 + h_2)_u = d_{bu} = (a_b + h)_u = h_u$ and

$d_{iu} = d_{3u} = (a_3 + h_3)_u = d_{tu} = (a_t + h_{ft})_u = h_u$ (Fig. 1).

$d = h - a_{st}$.

3.1. I-cross-section flexural members (here symbols shown in Fig. 3 are used)

In this case three rectangular elements of cross-section are involved. Thus coefficients of three Eqs. (5)-(7) are such:

$$\kappa_{n2} = (\omega_{c2} - \omega_{c1})b_c + (\omega_{b2} - \omega_{b1})b + (\omega_{t2} - \omega_{t1})b_t \,;$$

$$\kappa_{n1} = 2\omega_{c2}b_c h_{fcu} + 2\omega_{b2}bh_u + 2b_t(\omega_{t2}h_u - \omega_{t1}a_{tu}) + k_{st}\alpha_{est}\nu_{St}A_{st} + k_{sc}\alpha_{esc}\nu_{Sc}A_{sc} \,;$$

$$\kappa_{n0} = \omega_{c2}b_c h_{fcu}^2 + \omega_{b2}bh_u^2 + b_t(\omega_{t2}h_u^2 - \omega_{t1}a_{tu}^2) + k_{st}\alpha_{est}\nu_{St}A_{st}d + k_{sc}\alpha_{esc}\nu_{Sc}A_{sc}a_{scu} \,;$$

$$\kappa_n = P_t\nu_{St}/\nu_{pt} + P_c\nu_{Sc}/\nu_{pc} + N \,;$$

$$\kappa_{m3} = [(\varpi_{c2} - \varpi_{c1}) - (\omega_{c2} - \omega_{c1})]b_c + [(\varpi_{b2} - \varpi_{b1}) - (\omega_{b2} - \omega_{b1})]b + [(\varpi_{t2} - \varpi_{t1}) - (\omega_{t2} - \omega_{t1})]b_t \,;$$

$$\kappa_{m2} = (3\varpi_{c2} - 2\omega_{c2})b_c h_{fcu} + (3\varpi_{b2} - 2\omega_{b2})bh_u + b_t[3(\varpi_{t2}h_u - \varpi_{t1}a_{tu}) - 2(\omega_{t2}h_u - \omega_{t1}a_{tu})] \,;$$

$$\kappa_{m1} = (3\varpi_{c2} - \omega_{c2})b_c h_{fcu}^2 + (3\varpi_{b2} - \omega_{b2})bh_u^2 + b_t\left[3(\varpi_{t2}h_u^2 - \varpi_{t1}a_{tu}^2) - (\omega_{t2}h_u^2 - \omega_{t1}a_{tu}^2)\right] + $$
$$+ k_{st}\alpha_{est}\nu_{St}A_{st}d + k_{sc}\alpha_{esc}\nu_{Sc}A_{sc}a_{scu} \,;$$

$$\kappa_{m0} = \varpi_{c2}b_c h_{fcu}^3 + \varpi_{b2}bh_u^3 + b_t(\varpi_{t2}h_u^3 - \varpi_{t1}a_{tu}^3) + k_{st}\alpha_{est}\nu_{St}A_{st}d^2 + k_{sc}\alpha_{esc}\nu_{Sc}A_{sc}a_{scu}^2 \,;$$

$$\kappa_m = (P_t\nu_{St}/\nu_{pt})d + (P_c\nu_{Sc}/\nu_{pc})a_{scu} + Ne - M \,.$$

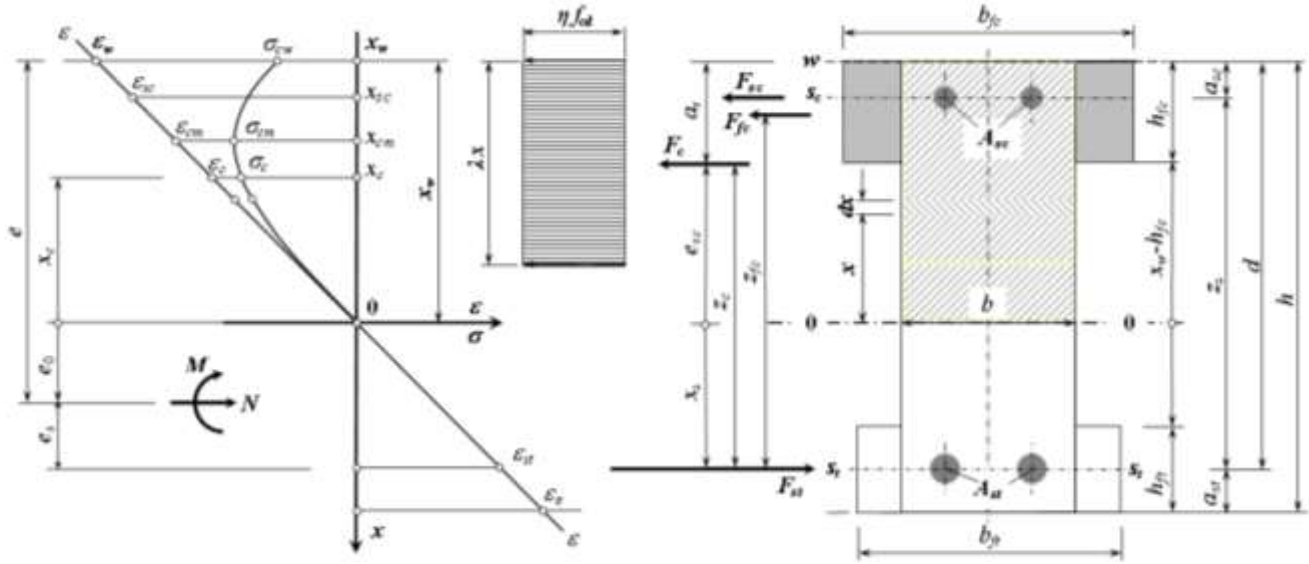

Fig. 3 Cross-section of flexural member, its acting forces and rectangular stress distribution in concrete compression zone of beams presented in EN-2 regulations

When there is no flange in tension then in formulae of coefficients κ_{nk} and κ_{mk} value of $b_t = 0$. If there is no compression flange then value of $b_c = 0$. For the case of rectangular cross-section both, $b_t = 0$ and $b_c = 0$. Letter k indicates coefficient number.

3.2. Flexural members of rectangular cross-section

$$\kappa_{n2} = (\omega_{b2} - \omega_{b1})b \,;$$

$$\kappa_{n1} = 2\omega_{b2}bh_u + k_{st}\alpha_{est}A_{st}\nu_{St} + k_{sc}\alpha_{esc}A_{sc}\nu_{Sc} \,;$$

$$\kappa_{n0} = \omega_{b2}bh_u^2 + k_{st}\alpha_{est}A_{st}\nu_{St}d + k_{sc}\alpha_{esc}A_{sc}\nu_{Sc}a_{scu} \,;$$

$$\kappa_n = P_t\nu_{St}/\nu_{pt} + P_c\nu_{Sc}/\nu_{pc} + N \,;$$

$$\kappa_{m3} = [(\varpi_{b2} - \varpi_{b1}) - (\omega_{b2} - \omega_{b1})]b \,;$$

$$\kappa_{m2} = (3\varpi_{b2} - 2\omega_{b2})bh_u \,;$$

$$\kappa_{m1} = (3\varpi_{b2} - \omega_{b2})bh_u^2 + k_{st}\alpha_{est}A_{st}\nu_{St}d + k_{sc}\alpha_{esc}A_{sc}\nu_{Sc}a_{scu} \,;$$

$$\kappa_{m0} = \varpi_{b2}bh_u^3 + k_{st}\alpha_{est}A_{st}\nu_{St}d^2 + k_{sc}\alpha_{esc}A_{sc}\nu_{Sc}a_{scu}^2 \,;$$

$$\kappa_m = (P_t\nu_{St}/\nu_{pt})d + (P_c\nu_{Sc}/\nu_{pc})a_{scu} + Ne - M \,.$$

3.3. Flexural members of rectangular cross-section with reinforcement in tensile zone only, when $A_{st} = \rho_l bd$, values of coefficients reduced by b times

$$\kappa_{n2} = (\omega_{b2} - \omega_{b1}) \,;$$

$$\kappa_{n1} = 2\omega_{b2}h_u + k_{st}\alpha_{est}\nu_{St}\rho_l d \,;$$

$$\kappa_{n0} = \omega_{b2}h_u^2 + k_{st}\alpha_{est}\nu_{St}\rho_l d^2 \,;$$

$$\kappa_n = \frac{P_t\nu_{St}/\nu_{pt} + N}{b} \,;$$

$$\kappa_{m3} = (\varpi_{b2} - \varpi_{b1}) - (\omega_{b2} - \omega_{b1}) \,;$$

$$\kappa_{m2} = (3\varpi_{b2} - 2\omega_{b2})h_u \,;$$

$$\kappa_{m1} = (3\varpi_{b2} - \omega_{b2})h_u^2 + k_{st}\alpha_{est}\rho_l\nu_{St}d^2 \,;$$

$$\kappa_{m0} = \varpi_{b2}h_u^3 + k_{st}\alpha_{est}\rho_l\nu_{St}d^3 \,;$$

$$\kappa_m = \frac{(P_t\nu_{St}/\nu_{pt})d + Ne - M}{b} \,.$$

3.4. T-cross-section flexural members with flange in compression zone only and with reinforcement in both tension and compression zones. Tensile flange and web in tension zone are neglected

$$\kappa_{n2} = (\omega_{c2} - \omega_{c1})b_c - \omega_{b1}b;$$

$$\kappa_{n1} = 2\omega_{c2}b_c h_{fcu} + k_{st}\alpha_{est}\nu_{St}A_{st} + k_{sc}\alpha_{esc}\nu_{Sc}A_{sc};$$

$$\kappa_{n0} = \omega_{c2}b_c h_{fcu}^2 + k_{st}\alpha_{est}\nu_{St}A_{st}d + k_{sc}\alpha_{esc}\nu_{Sc}A_{sc}a_{scu};$$

$$\kappa_n = P_t\nu_{St}/\nu_{pt} + P_c\nu_{Sc}/\nu_{pc} + N;$$

$$\kappa_{m3} = [(\omega_{c1} - \varpi_{c1}) - (\omega_{c2} - \varpi_{c2})]b_c + (\omega_{b1} - \varpi_{b1})b;$$

$$\kappa_{m2} = (3\varpi_{c2} - 2\omega_{c2})b_c h_{fcu};$$

$$\kappa_{m1} = (3\varpi_{c2} - \omega_{c2})b_c h_{fcu}^2 + k_{st}\alpha_{est}\nu_{St}A_{st}d + k_{sc}\alpha_{esc}\nu_{Sc}A_{sc}a_{scu};$$

$$\kappa_{m0} = \varpi_{c2}b_c h_{fcu}^3 + k_{st}\alpha_{est}\nu_{St}A_{st}d^2 + k_{sc}\alpha_{esc}\nu_{Sc}A_{sc}a_{scu}^2;$$

$$\kappa_m = (P_t\nu_{St}/\nu_{pt})d + (P_c\nu_{Sc}/\nu_{pc})a_{scu} + Ne - M.$$

3.5. Rectangular cross-section flexural members with reinforcement in tension zone only when $k_{st} = 1$ and $\nu_{St} = 1$. Tension zone of web is neglected

$$\kappa_{m3} = (\omega_{b1} - \varpi_{b1})b;$$

$$\kappa_{n2} = -\omega_{b1}b; \qquad \kappa_{m2} = 0;$$

$$\kappa_{n1} = \alpha_{est}A_{st}; \qquad \kappa_{m1} = \alpha_{est}A_{st}d = \kappa_{n0};$$

$$\kappa_{n0} = \alpha_{est}A_{st}d = \kappa_{n1}d; \qquad \kappa_{m0} = \alpha_{est}A_{st}d^2 = \kappa_{m1}d;$$

$$\kappa_n = P_t + N; \qquad \kappa_m = P_t d + Ne - M.$$

For this case Eq. (7) can be presented in the following shape as well:

$$q_3x_w^3 + q_2x_w^2 + q_1x_w + q_0 = 0, \tag{13}$$

where

$$q_3 = (\omega_{b1} - \varpi_{b1})bN;$$

$$q_2 = \omega_{b1}b(Ne - M);$$

$$q_1 = \alpha_{est}A_{st}dN - \alpha_{est}A_{st}(Ne - M) = = \alpha_{est}A_{st}[N(d-e) + M];$$

$$q_0 = \alpha_{est}A_{st}d^2N - \alpha_{est}A_{st}d(Ne - M) = = \alpha_{est}A_{st}d[N(d-e) + M] = q_1d.$$

If $N = 0$ then dividing by M gives:

$$q_3 = 0;$$

$$q_2 = -\omega_{b1}bM/M = -b\omega_{b1};$$

$$q_1 = \alpha_{est}A_{st}M/M = \alpha_{est}A_{st};$$

$$q_0 = \alpha_{est}A_{st}dM/M = \alpha_{est}A_{st}d = q_1d.$$

Notation $A_{st} = \rho_l bd$ allows writing:

$$q_2 = -\omega_{b1};$$

$$q_1 = \alpha_{est}d\rho_l;$$

$$q_0 = \alpha_{est}d^2\rho_l = q_1d,$$

or

$$\left.\begin{aligned} x_w^2 - sdx_w - sd^2 &= 0; \\ \xi_w^2 - s\xi_w - s &= 0, \end{aligned}\right\} \tag{14}$$

where $s = \dfrac{\alpha_{est}\rho_l}{\omega_{b1}}$; $x_w = \xi_w d$.

When x_w is calculated from (7), (13) or (14) then ε_w is calculated from (5) or (6), from (9) is obtained ω_i and the cycle of approximation is repeated until desired difference between values of ε_w obtained at the start and the end of the cycle is attained.

Attribution of flange forces to external forces (in analogous way as prestressing force P is attributed) enables to simplify application of formulae.

4. Main advantages and application possibilities of ZI method

1. Equations of ZI method are theoretical ones. Tests are required only for determination of $\sigma - \varepsilon$ diagrams of materials and verification of theoretical conclusions;

2. value of each parameter of cross-section is calculated separately but not total values of two or several parameters;

3. real values of parameters are calculated but not arbitrary ones;

4. ZI method is fairly general one:

1) it is applicable for calculation of any cross-section members which are symmetrical in respect to the plane passing via longitudinal axes of the member and bending moments M and/or forces N act in the said plane;

2) neutral axes can be located both within the cross-section and outside it. In the latter case each value of ω_i, ϖ_i and other parameters for the case of compression are determined for materials under compression, in the case of tension – for materials under tension;

3) member can be layered;

4) it is applicable for any load level (the same three equations of static equilibrium are applied from the beginning of member loading up to its failure);

5) it is suitable for any material (reinforced concrete, concrete, steel, timber, plastics etc.);

6) axial (longitudinal) reinforcing may be of any type (reinforcement may be prestressed or not prestressed, or mixed, it may be located in any layer of the member). It is possible to calculate value of reinforcement prestressing at which failure of member is initiated by yield of tensile prestressed reinforcement when reinforcement stress reaches yield limit of the steel;

7) it may be applied for calculation of cross-sections for members without cracks and for calculation of cross-sections at cracks and these between cracks of members with cracks;

8) intensity of reinforcing may be any, i. e. calcu-

lation of normally and abundantly reinforced members is possible. Strength calculation of members reinforced with not prestressed high strength reinforcement in the cases when member fails before reinforcement stress reaches yield limit. It may be relevant in cases when end pieces of reinforcement remaining during manufacture of prestressed structures are used as reinforcement of non prestressed concrete structures;

9) calculation of stress-strain state in plastic hinges of continuous members is possible, i. e. when carrying capacity reduces because reinforcement stresses exceed yield limit and/or reduction of concrete compression stresses begins;

10) various, not only curvilinear, stress diagrams are possible or stresses can be neglected as well. Ability to change stress-strain diagram shape gives opportunity for investigation in influence of scale factor and strain gradient of member layers on stress-strain state parameters [19, 20];

11) evaluation of member layers strain declination from these values corresponding the plane sections is possible. It enables to take into account slip of member layers in respect to each other. For example, the method can be used for calculation of members made of timber bars joint by keys or pins. Slip value in this case has to be given. Theoretical analyses of slip effect on other stress-strain parameters, e. g. deflection, can be performed;

5. ZI method is based on fundamentals of technical subjects lectured for engineering students but instead of equations for elastic materials equations developed for elastic plastic materials are used. Equations of ZI method shall be considered as extension of application area of widely used formulas for elastic materials – they are applicable not only for elastic materials but and for elastic plastic materials as well. When values of some coefficients of ZI method equations (unities, zeroes) are inscribed, in particular cases formulae for elastic materials are obtained. ZI method being quite general (it allows solution of many problems) is suitably simple – it can be understood and used by persons having fundamental knowledge of technical subjects (mechanics, strength of materials and structural analyses) delivered at universities;

6. in particular but quite frequent cases from general equations, which in general case are solved by the method of successive approximation, directly calculated (without successive approximations) equations can be obtained [12-14];

7. ZI method can be applied for solution of inverse problem when stress-strain state of member is to be determined using experimental parameters of normal cracks (depth, width and distance between the cracks) [21];

8. ZI method may be used as a reference for development and estimation of approximate calculation methods.

5. Conclusions

It may be considered that creation of ZI method fundamentals is finished by the third Eq. (7) for equilibrium of forces.

Using three equation system of ZI method many problems of theory and practice can be solved (see sections 3 and 4).

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I. Židonis

TREČIOJI LENKIAMŲ ELEMENTŲ SKERSPJŪVIO JĖGŲ PUSIAUSVYROS LYGTIS

R e z i u m ė

Ankstesniuose šio straipsnio autoriaus darbuose pateiktos gana universalaus ZI metodo dvi jėgų pusiausvyros lygtys realioms įtempių-deformacijų būvio parametrų reikšmėms apskaičiuoti statmename lenkiamų elementų ašiai skerspjūvyje bet kurioje apkrovimo stadijoje. Reikėjo iš anksto žinoti arba nuoseklaus artėjimo būdu apskaičiuoti kurio nors vieno sluoksnio deformaciją. Šiame straipsnyje pateikiama trečia jėgų statinės pusiausvyros lygtis. Pagrindinė lygties paskirtis – apskaičiuoti lenkiamo elemento neutraliosios ašies padėtį atvejais, kai žinomi išoriniai poveikiai, bet iš anksto nežinoma nei vieno sluoksnio deformacija. Iš kitų pusiausvyros lygčių apskaičiuojama gniuždomos zonos krašto ε_w deformacija, o tai jau leidžia apskaičiuoti ir visų kitų parametrų reikšmes. Toks skaičiavimas ypatingai aktualus gelžbetoniniams elementams su plyšiais tempiamoje zonoje, nes šiuo metu neutraliosios ašies padėčiai apskaičiuoti naudojama arba empirinė formulė, arba skaičiuojama labai apytiksliai.

Trečiąja jėgų pusiausvyros lygtimi, galima laikyti, užbaigiamas ZI metodo pagrindo sukūrimas. Panaudojant ZI metodo trijų lygčių sistemą, galima spręsti daugybę įvairių teorinių ir praktinių uždavinių. Šiame straipsnyje taip pat pateikiama ZI metodo bendrojo atvejo trijų lygčių pritaikymo konkretiems pagrindiniams praktiniams atvejams pavyzdžių.

I. Židonis

THE THIRD EQUILIBRIUM EQUATION FOR FORCES OF FLEXURAL MEMBER CROSS-SECTION

S u m m a r y

In earlier articles of the author two equilibrium equations of fairly general ZI method for calculation of real parameter values of stress-strain states for normal to longitudinal axes sections of flexural members at any loading stage are presented. It required in advance to have or to calculate by the method of successive approximation strain value of any one layer. The third equation for force static is presented in this article. The main purpose of the equation is calculation of neutral axes location in cases when external actions are given but strain value of any layer is not given in advance. Strain ε_w is calculated from other equations and then it enables calculation of values for all other parameters. Such calculation is especially relevant for reinforced concrete members with cracks in tension zone because at present calculation of neutral axes location is carried out either by empirical formula or in very approximate way.

It can be considered that creation of ZI method is finished by the third force equilibrium equation. Application of ZI method three equation system enables solution of many theoretical and practical problems. Application for particular practical problems examples of ZI method three equations in general case are presented in this article as well.

Keywords: third equilibrium equation, forces, flexural member, cross-section.

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