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Analysis of nonlinear vibration of coupled systems with cubic nonlinearity

M. Bayat*, I. Pakar, M. Shahidi*****
**Department of Civil Engineering, Torbateheydarieh Branch, Islamic Azad University, Torbateheydarieh, Iran*
E-mail: mbayat14@yahoo.com
***Department of Civil Engineering, Torbateheydarieh Branch, Islamic Azad University, Torbateheydarieh, Iran*
E-mail: iman.pakar@yahoo.com
****Department of Civil Engineering, Torbateheydarieh Branch, Islamic Azad University, Torbateheydarieh, Iran*
E-mail: mehran.shahidi13@gmail.com

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1. Introduction

In the past few decades the motion of multidegree of freedom (multi-DOF) oscillation systems has been widely considered. Moochhala and Raynor [1] proposed an approximate method for the motions of unequal masses connected by (n+1) nonlinear springs and anchored to rigid end walls. Huang [2] studied on the Harmonic oscillations of nonlinear two-degree-of-freedom systems. Gilchrist [3] analyzed the free oscillations of conservative quasilinear systems with two degrees of freedom. Efstathiades [4] developed the work on the existence and characteristic behaviour of combination tones in nonlinear systems with two degrees of freedom. Alexander and Richard [5] considered the resonant dynamics of a two-degree-of-freedom system composed of a linear oscillator weakly coupled to a strongly nonlinear one, with an essential (nonlinearizable) cubic stiffness nonlinearity. Chen [6] used generalized Galerkin's method to nonlinear oscillations of two-degree-of-freedom systems. Ladygina and Manevich [7] investigated the free oscillations of a conservative system with two degrees of freedom having cubic nonlinearities (of symmetric nature) and close natural frequencies by using multiscale method. Cveticanin [8, 9] used a combination of a Jacobi elliptic function and a trigonometric function to obtain an analytical solution for the motion of a two-mass system with two degrees of freedom in which the masses were connected with three springs.

Two degree of freedom (TDOF) systems are very important in physics and engineering and many practical engineering vibration systems such as elastic beams supported by two springs and vibration of a milling machine [10] can be studied by considering them as a TDOF systems. The TDOF oscillation systems consist of two second-order differential equations with cubic nonlinearities. So, a set of differential algebraic equations by introducing new variables was obtained from transforming the equations of motion of a mechanical system which associated with the linear and nonlinear springs. In general, finding an exact analytical solution for nonlinear equations is extremely difficult. Therefore, many analytical and numerical approaches have been investigated. The most useful methods for solving nonlinear equations are perturbation methods. They are not valid for strongly nonlinear equations and there have many shortcomings. Many new techniques have appeared in the open literature to overcome the shortcomings, such as Homotopy perturbation [11], energy balance [12-15], variational approach [16, 17], max-min approach [18], Iteration perturbation method [19] and other analytical and numerical methods [20-32].

In the present paper, we applied He's Max-Min Approach (MMA) and He's Improved Amplitude-Frequency Formulation (IAFF) for nonlinear oscillators which were proposed by J.H. He [26, 30]. Both of them lead us to a very rapid convergence of the solution, and they can be easily extended to other nonlinear oscillations. Comparisons between analytical and exact solutions show that He's MMA and He's IAFF methods can converge to an accurate periodic solution for nonlinear systems.

2. Basic idea of he's max-min approach method

We consider a generalized nonlinear oscillator in the form

$$u'' + u\,f(u) = 0, \qquad u(0) = A, \qquad u'(0) = 0 \tag{1}$$

where $f(u)$ is a nonnegative function of u. According to the idea of the max-min method, we choose a trial function in the form

$$u(t) = A\cos(\omega t) \tag{2}$$

where the ω unknown frequency to be further is determined. Observe that the square of frequency, ω^2, is never less than that in the solution

$$u_1(t) = A\cos\left(\sqrt{f_{min}}\, t\right) \tag{3}$$

of the following linear oscillator

$$u'' + u\,f_{min} = 0, \quad u(0) = A, \quad u'(0) = 0 \tag{4}$$

where f_{min} is the minimum value of the function $f(u)$. In addition, ω^2 never exceeds the square of frequency of the solution

$$u_1(t) = A\cos\left(\sqrt{f_{max}}\, t\right) \tag{5}$$

of the following oscillator

$$u'' + u\,f_{max} = 0, \quad u(0) = A, \quad u'(0) = 0 \tag{6}$$

where f_{max} is the maximum value of the function $f(u)$.

Hence, it follows that

$$\frac{f_{min}}{1} < \omega^2 < \frac{f_{max}}{1} \tag{7}$$

According to He Chentian interpolation [26, 27], we obtain

$$\omega^2 = \frac{m\, f_{min} + n\, f_{max}}{m+n} \tag{8}$$

or

$$\omega^2 = \frac{f_{min} + k\, f_{max}}{1+k} \tag{9}$$

where m and n are weighting factors, $k = n/m$. So the solution of Eq. (1) can be expressed as

$$u(t) = A\cos\sqrt{\frac{f_{min} + k\, f_{max}}{1+k}}\, t \tag{10}$$

The value of k can be approximately determined by various approximate methods [26-28]. Among others, hereby we use the residual method [26]. Substituting Eq. (10) into Eq. (1) results in the following residual

$$\begin{aligned} R(t;k) = -\omega^2 A\cos(\omega t) + \\ +\left(A\cos(\omega t)\right) f\left(A\cos(\omega t)\right) \end{aligned} \tag{11}$$

where $\omega = \sqrt{\frac{f_{min} + k\, f_{max}}{1+k}}$, if by chance, Eq. (10) is the exact solution, then $R(t;k)$ is vanishing completely. Since our approach is only an approximation to the exact solution, we set

$$\int_0^T R(t;k)\cos\sqrt{\frac{f_{min} + k\, f_{max}}{1+k}}\, t\, dt = 0 \tag{12}$$

where $T = 2\pi/\omega$. Solving the above equation, we can easily obtain.

In the present paper, we consider a general nonlinear oscillator in the form [29]

$$k = \frac{f_{max} - f_{min}}{1 - \sqrt{\frac{A}{\pi}\int_0^\pi \cos^2 x\, f\left(A\cos x\right) dx}} \tag{13}$$

Substituting the above equation into Eq. (10), we obtain the approximate solution of Eq. (1).

3. Basic idea of improved amplitude-frequency formulation

We consider a generalized nonlinear oscillator in the form [30]

$$u'' + f(u) = 0,\ \ u(0) = A,\ \ u'(0) = 0 \tag{14}$$

We use two following trial functions

$$u_1(t) = A\cos(\omega_1 t) \tag{15}$$

and

$$u_2(t) = A\cos(\omega_2 t) \tag{16}$$

The residuals are

$$R_1(\omega t) = -A\omega_1^2 \cos(\omega_1 t) + f\left(A\cos(\omega_1 t)\right) \tag{17}$$

and

$$R_2(\omega t) = -A\omega_2^2 \cos(\omega_2 t) + f\left(A\cos(\omega_2 t)\right) \tag{18}$$

The original frequency-amplitude formulation reads [30, 31]

$$\omega^2 = \frac{\omega_1^2 R_2 - \omega_2^2 R_1}{R_2 - R_1} \tag{19}$$

He used the following formulation [30, 31] and Geng and Cai improved the formulation by choosing another location point [31].

$$\omega^2 = \frac{\omega_1^2 R_2\ (\omega_2 t = 0) - \omega_2^2 R_1\ (\omega_1 t = 0)}{R_2 - R_1} \tag{20}$$

This is the improved form by Geng and Cai.

$$\omega^2 = \frac{\omega_1^2 R_2\ (\omega_2 t = \pi/3) - \omega_2^2 R_1\ (\omega_1 t = \pi/3)}{R_2 - R_1} \tag{21}$$

The point is: $\cos(\omega_1 t) = \cos(\omega_2 t) = k$.

Substituting the obtained ω into $u(t) = A\cos(\omega t)$, we can obtain the constant k in ω^2 equation in order to have the frequency without irrelevant parameter.

To improve its accuracy, we can use the following trial function when they are required

$$\left.\begin{aligned} u_1(t) &= \sum_{i=1}^{m} A_i \cos(\omega_i t) \\ u_2(t) &= \sum_{i=1}^{m} A_i \cos(\Omega_i t) \end{aligned}\right\} \tag{22}$$

or

$$\left.\begin{aligned} u_1(t) &= \frac{\sum_{i=1}^{m} A_i \cos(\omega_i t)}{\sum_{j=1}^{m} B_j \cos(\omega_j t)} \\ u_2(t) &= \frac{\sum_{i=1}^{m} A_i \cos(\Omega_i t)}{\sum_{j=1}^{m} B_j \cos(\Omega_j t)} \end{aligned}\right\} \tag{23}$$

But in most cases because of the sufficient accuracy, trial functions are as follow and just the first term

$$\left.\begin{aligned} u_1(t) &= A\cos t \\ u_2(t) &= a\cos(\omega\, t) + (A-a)\cos(\omega\, t) \end{aligned}\right\} \tag{24}$$

and

$$\left.\begin{aligned} u_1(t) &= A\cos t \\ u_2(t) &= \frac{A(1+c)\cos(\omega\, t)}{1+c\cos(2\omega\, t)} \end{aligned}\right\} \tag{25}$$

where a and c are unknown constants. In addition we can set $\cos t = k$ in u_1, and $\cos(\omega\, t) = k$ in u_2.

4. Examples of nonlinear two degree of freedom (TDOF) oscillating systems

In this section, two practical examples of TDOF oscillation systems are illustrated to show the applicability, accuracy and effectiveness of the proposed approach.

4.1. Example 1

A two-mass system connected with linear and nonlinear stiffnesses. Consider the two-mass system model as shown in Fig. 1. The equation of motion is given as [9]

$$\left.\begin{aligned} m\ddot{x} + k_1(x-y) + k_2(x-y)^3 &= 0 \\ m\ddot{y} + k_1(y-x) + k_2(y-x)^3 &= 0 \end{aligned}\right\} \tag{26}$$

with initial conditions

$$\left.\begin{aligned} x(0) &= X_0, \quad \dot{x}(0) = 0 \\ y(0) &= Y_0, \quad \dot{y}(0) = 0 \end{aligned}\right\} \tag{27}$$

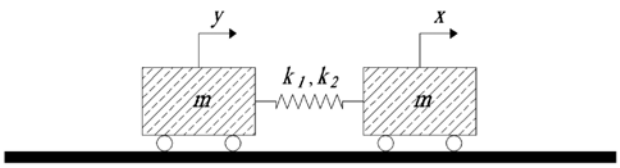

Fig. 1 Two masses connected by linear and nonlinear stiffnesses

Where double dots in Eq. (26) denote double differentiation with respect to time, k_1 and k_2 are linear and nonlinear coefficients of the spring stiffness, respectively. Dividing Eq. (26) by mass m yields

$$\left.\begin{aligned} \ddot{x} + \frac{k_1}{m}(x-y) + \frac{k_2}{m}(x-y)^3 &= 0 \\ \ddot{y} + \frac{k_1}{m}(y-x) + \frac{k_2}{m}(y-x)^3 &= 0 \end{aligned}\right\} \tag{28}$$

Introducing intermediate variables u and v as follows [32]

$$x := u \tag{29a}$$

$$y - x := v \tag{29b}$$

and transforming Eqs. (29.a) and (29.b) yields

$$\ddot{u} - \alpha v - \beta v^3 = 0 \tag{30}$$

$$\ddot{v} + \ddot{u} + \alpha v + \beta v^3 = 0 \tag{31}$$

where $\alpha = k_1/m$ and $\beta = k_2/m$. Eq. (30) is rearranged as follows

$$\ddot{u} = \alpha v + \beta v^3 \tag{32}$$

Substituting Eq. (32) into Eq. (31) yields

$$\ddot{v} + 2\alpha v + 2\beta v^3 = 0 \tag{33}$$

with initial conditions

$$v(0) = y(0) - x(0) = Y_0 - X_0 = A, \quad \dot{v}(0) = 0. \tag{34}$$

4.1.1. Solution using MMA

We can rewrite Eq. (32) in the following form

$$\ddot{v} + \left(2\alpha + 2\beta v^2\right)v = 0 \tag{35}$$

We choose a trial-function in the form

$$v = A\cos(\omega t) \tag{36}$$

where ω the frequency to be is determined the maximum and minimum values of $2\alpha + 2\beta v^2$ will be $2\alpha + 2\beta A^2$ and 2α respectively, so we can write

$$\frac{2\alpha}{1} < \omega^2 = 2\alpha + 2\beta v^2 < \frac{2\alpha + 2\beta A^2}{1} \tag{37}$$

According to He Chengtian's inequality [27, 28], we have

$$\omega^2 = \frac{m2\alpha + n\left(2\alpha + 2\beta A^2\right)}{m+n} = 2\alpha + 2k\beta A^2 \tag{38}$$

where m and n are weighting factors, $k = n/m+n$. Therefore the frequency can be approximated as

$$\omega = \sqrt{2\alpha + 2k\beta A^2} \tag{39}$$

Its approximate solution reads

$$v = A\cos\sqrt{2\alpha + 2k\beta A^2}\, t \tag{40}$$

In view of the approximate solution, Eq. (40), we rewrite Eq. (33) in the form

$$\ddot{v} + \left(2\alpha + 2k\beta A^2\right)v = \left(2\alpha + 2k\beta A^2\right)v - 2\beta v^3 \tag{41}$$

If by any chance Eq. (30) is the exact solution, then the right side of Eq. (31) vanishes completely. Considering our approach which is just an approximation one, we set

$$\int_0^{T/4}\left(2k\beta A^2 v-2\beta v^3\right)\cos\omega t\,dt=0 \tag{42}$$

where $T=2\pi/\omega$. Solving the above equation, we can easily obtain

$$k=\frac{3}{4} \tag{43}$$

Finally the frequency is obtained as

$$\omega=\frac{1}{2}\sqrt{8\alpha+6\beta A^2} \tag{44}$$

According to Eqs. (36) and (44), we can obtain the following approximate solution

$$v(t)=A\cos\left(\frac{1}{2}\sqrt{8\alpha+6\beta A^2}\,t\right) \tag{45}$$

The first-order analytical approximation for $u(t)$ is

$$u(t)=\iint\left(\alpha v+\beta v^3\right)dt\,dt = \\ =-\frac{1}{9\omega^2}A\cos(\omega t)\begin{pmatrix}9\alpha+6\beta A^2+\\ +A\beta\cos^2(\omega t)\end{pmatrix} \tag{46}$$

Therefore, the first-order analytical approximate displacements $x(t)$ and $y(t)$ are

$$\left.\begin{array}{l}x(t)=u(t)\\ x(t)=u(t)+A\cos(\omega t)\end{array}\right\}. \tag{47}$$

4.1.2. Solution using IAFF

We use trial functions, as follows:

$$v_1(t)=A\cos t \tag{48}$$

and

$$v_2(t)=A\cos(2t) \tag{49}$$

Respectively, the residual equations are

$$R_1(t)=A\cos(t)\left(-1+2\alpha+2\beta A^2\cos^2(t)\right) \tag{50}$$

and

$$R_2(t)=2A\cos(2t)\left(-2+\alpha+\beta A^2\cos^2(2t)\right) \tag{51}$$

Considering $\cos t_1=\cos 2t_2=k$ we have

$$\omega^2=\frac{\omega_1{}^2R_2-\omega_2{}^2R_1}{R_2-R_1}=2\alpha+2\beta k^2A^2 \tag{52}$$

We can rewrite $v(t)=A\cos(\omega t)$ in the form

$$v(t)=A\cos\left(\sqrt{2\alpha+2\beta k^2A^2}\,t\right) \tag{53}$$

In view of the approximate solution, we can rewrite the main equation in the form

$$\ddot{v}+\left(2\alpha+2\beta k^2A^2\right)v=\left(2\beta k^2A^2\right)v-2\beta v^3 \tag{54}$$

If by any chance $v(t)=A\cos\left(\sqrt{2\alpha+2\beta k^2A^2}\,t\right)$ is the exact solution, then the right side of Eq. (54) vanishes completely. Considering our approach which is just an approximation one, we set

$$\int_0^{T/4}\left(2\beta k^2A^2v-2\beta v^3\right)\cos\omega t\,dt=0\ ,T=2\pi/\omega \tag{55}$$

Considering $v(t)=A\cos\left(\sqrt{2\alpha+2\beta k^2A^2}\,t\right)$ and substituting to Eq. (55) and solving the integral t, we have

$$k=\frac{1}{2}\sqrt{3} \tag{56}$$

So, substituting Eq. (56) into Eq. (52), we have

$$\omega=\frac{1}{2}\sqrt{8\alpha+6\beta A^2} \tag{57}$$

4.2. Example 2

A two-mass system connected with linear and nonlinear stiffnesses fixed to the body.

Consider a two-mass system connected with linear and nonlinear springs and fixed to a body at two ends as shown in Fig. 2.

$$\left.\begin{array}{l}m\ddot{x}+k_1x+k_2(x-y)+k_3(x-y)^3=0\\ m\ddot{y}+k_1x+k_2(y-x)+k_3(y-x)^3=0\end{array}\right\} \tag{58}$$

with initial conditions

$$\left.\begin{array}{ll}x(0)=X_0, & \dot{x}(0)=0\\ y(0)=Y_0, & \dot{y}(0)=0\end{array}\right\} \tag{59}$$

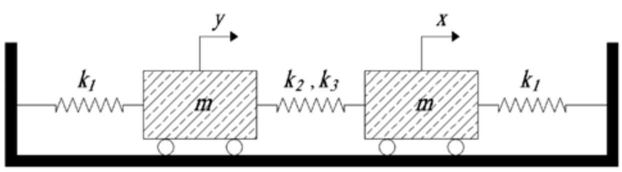

Fig. 2 Two-mass system connected with the fixed bodies

Where double dots in Eq. (58) denote double differentiation with respect to time *t*, k_1 and k_2 are linear and nonlinear coefficients of the spring stiffness and k_3 is the nonlinear coefficient of the spring stiffness. Dividing Eq. (58) by mass *m* yields

$$\left.\begin{aligned}\ddot{x}+\frac{k_1}{m}x+\frac{k_2}{m}(x-y)+\frac{k_3}{m}(x-y)^3=0\\ \ddot{y}+\frac{k_1}{m}x+\frac{k_2}{m}(y-x)+\frac{k_3}{m}(y-x)^3=0\end{aligned}\right\} \tag{60}$$

Like in Example 1, transforming the above equations using intermediate variables in Eqs. (29.a) and (29.b) yields

$$\ddot{u}+\alpha u-\beta v-\xi v^3=0 \tag{61}$$

$$\ddot{u}+\ddot{v}+\alpha u-\alpha v+\beta v+\xi v^3=0 \tag{62}$$

where $\alpha=k_1/m$, $\beta=k_2/m$ and $\xi=k_3/m$. Eq. (61) is rearranged as follows

$$\ddot{u}=-\alpha u+\beta v+\xi v^3 \tag{63}$$

Substituting Eq. (61) into Eq. (62) yields

$$\ddot{v}+(\alpha+2\beta)v+2\xi v^3=0 \tag{64}$$

with initial conditions

$$v(0)=y(0)-x(0)=Y_0-X_0=A,\ \dot{v}(0)=0 \tag{65}$$

4.2.1. Solution using MMA

We can re-write Eq. (64) in the following form

$$\ddot{v}+\left((\alpha+2\beta)+2\xi v^2\right)v=0 \tag{66}$$

We choose a trial-function in the form

$$v=A\cos(\omega t) \tag{67}$$

where ω the frequency to be is determined the maximum and minimum values of $\alpha+2\beta+2\xi v^2$ will be $\alpha+2\beta+2\xi A^2$ and $\alpha+2\beta$ respectively, so we can write

$$\frac{\alpha+2\beta}{1}<\omega^2=\alpha+2\beta+2\xi v^2<\frac{\alpha+2\beta+2\xi A^2}{1} \tag{68}$$

According to He Chengtian's inequality [27, 28], we have

$$\omega^2=\frac{m.(\alpha+2\beta)+n.(\alpha+2\beta+2\xi A^2)}{m+n}==\alpha+2\beta+2\xi kA^2 \tag{69}$$

where m and n are weighting factors, $k=n/m+n$. Therefore the frequency can be approximated as

$$\omega=\sqrt{\alpha+2\beta+2\xi kA^2} \tag{70}$$

Its approximate solution reads

$$v=A\cos\sqrt{\alpha+2\beta+2\xi kA^2}\,t \tag{71}$$

In view of the approximate solution, Eq. (71), we re-write Eq. (64) in the form

$$\ddot{v}+\left(\alpha+2\beta+2\xi kA^2\right)v=\left(2\xi kA^2\right)v-2\xi v^3 \tag{72}$$

If by any chance Eq. (71) is the exact solution, then the right side of Eq. (72) vanishes completely. Considering our approach which is just an approximation one, we set

$$\int_0^{T/4}\left(2\xi kA^2v-2\xi v^3\right)\cos\omega t\,dt=0 \tag{73}$$

where $T=2\pi/\omega$. Solving the above equation, we can easily obtain

$$k=\frac{3}{4} \tag{74}$$

Finally the frequency is obtained as

$$\omega=\frac{1}{2}\sqrt{4\alpha+8\beta+6\xi A^2} \tag{75}$$

According to Eqs. (75) and (67), we can obtain the following approximate solution

$$v(t)=A\cos\left(\frac{1}{2}\sqrt{4\alpha+8\beta+6\xi A^2}\,t\right) \tag{76}$$

The first-order analytical approximation for $u(t)$ is

$$u(t)=\frac{-\cos\left(\sqrt{\alpha t}\right)\begin{pmatrix}-X_0\alpha^2+10X_0\alpha\omega^2-9X_0\omega^4+\\+\xi A^3\alpha-7\xi A^3\omega^2-9A\beta\omega^2+A\alpha\beta\end{pmatrix}}{\alpha^2-10\alpha\omega^2+9\omega^4}-\frac{27A\begin{pmatrix}\cos(\omega t)\left(\left(\xi A^2+\frac{4}{3}\beta\right)\left(\omega^2-\frac{1}{9}\alpha\right)\right)+\\+\cos(3\omega t)\left(\frac{1}{27}\xi A^2\left(\omega^2-\alpha\right)\right)\end{pmatrix}}{4\alpha^2-40\alpha\omega^2+36\omega^4} \tag{77}$$

Therefore, the first-order analytical approximate displacements $x(t)$ and $y(t)$ are

$$\left.\begin{aligned}x(t)&=u(t)\\ x(t)&=u(t)+A\cos(\omega t)\end{aligned}\right\}. \tag{78}$$

4. 2.2. Solution using IAFF

We use trial functions, as follows

$$v_1(t)=A\cos t \tag{79}$$

and

$$v_2(t)=A\cos(2t) \tag{80}$$

Respectively, the residual equations are

$$R_1(t) = A\cos(t)\left(-1+\alpha+2\beta+2\xi A^2\cos^2(t)\right) \qquad (81)$$

and

$$R_2(t) = A\cos(2t)\left(-4+\alpha+2\beta+2\xi A^2\cos^2(2t)\right) \qquad (82)$$

Considering $\cos\, t_1 = \cos\, 2t_2 = k$ we have

$$\omega^2 = \frac{\omega_1{}^2 R_2 - \omega_2{}^2 R_1}{R_2 - R_1} = \alpha + 2\beta + 2\xi k^2 A^2 \qquad (83)$$

We can rewrite $v(t) = A\cos(\omega t)$ in the form

$$v(t) = A\cos\left(\sqrt{\alpha+2\beta+2\xi k^2 A^2}\, t\right) \qquad (84)$$

In view of the approximate solution, we can rewrite the main equation in the form

$$\ddot{v} + \left(\alpha+2\beta+2\xi k^2 A^2\right)v = \left(2\xi k^2 A^2\right)v - 2\xi v^3 \qquad (85)$$

If by any chance Eq. (84) is the exact solution, then the right side of Eq. (85) vanishes completely. Considering our approach which is just an approximation one, we set

$$\int_0^{T/4}\left(2\xi k^2 A^2 v - 2\xi v^3\right)\cos\omega t\, dt = 0\ ,\ T = 2\pi/\omega \qquad (86)$$

Considering the term $v(t) = A\cos\left(\sqrt{2\alpha+2\beta k^2 A^2}\, t\right)$ and substituting the term to Eq. (86) and solving the integral term, we have

$$k = \frac{1}{2}\sqrt{3} \qquad (87)$$

So, substituting Eq. (87) into Eq. (86), we have

$$\omega = \frac{1}{2}\sqrt{4\alpha+8\beta+6\xi A^2}\ . \qquad (88)$$

5. Discussion of the examples

Comparisons with published data and exact solutions [8, 9] are presented and tabulated to illustrate and verify the accuracy of the MMA and IAFF .The first-order approximate solutions is of a high accuracy and the percentage error improves significantly from lower order to higher order analytical approximations for different parameters and initial amplitudes. Hence, it is concluded that excellent agreement with the exact so.

Table 1

Comparison of frequency corresponding to various parameters of system

Constant parameters					Analytical solution	Exact solution	Relative error %
m	k_1	k_2	X_0	Y_0	$\omega_{MMA=IAFF}$	ω_{Exact} [9]	$\left(\omega_{MMA=IAFF} - \omega_{Ex}\right)/\omega_{Ex}$
1	0.5	0.5	1	5	3.605551	3.539243	1.873506
1	1	1	5	1	5.09902	5.005246	1.873506
5	2	0.5	5	10	4.421538	4.333499	2.031592
10	5	5	10	20	8.717798	8.533586	2.158667
20	40	50	20	10	19.46792	19.05429	2.170820
50	100	50	-10	20	36.79674	36.00234	2.206522
100	400	100	50	-50	122.5071	119.8489	2.217969
200	100	50	100	300	183.7145	179.7239	2.220354
500	500	1000	400	600	244.9571	239.6368	2.220179

Table 2

Comparison of frequency corresponding to various parameters of system

Constant parameters						Analytical solution	Exact solution	Relative error %
m	k_1	k_2	k_3	X_0	Y_0	$\omega_{MMA=IAFF}$	ω_{Exact} [7]	$\left(\omega_{MMA=IAFF} - \omega_{Ex}\right)/\omega_{Ex}$
1	0.5	0.5	0.5	1	5	3.674235	3.611743	1.730234
1	1	1	2	5	1	7.141428	7.004694	1.952045
5	2	0.5	5	5	10	6.17252	6.042804	2.146618
10	5	5	10	10	20	12.30853	12.04665	2.173874
20	40	50	50	20	10	19.54482	19.13632	2.134672
50	100	50	100	-10	20	52.00000	50.87391	2.213492
100	400	100	200	50	-50	173.2224	169.4611	2.219547
200	100	50	400	200	300	244.951	239.6302	2.220415
500	500	1000	500	400	600	346.4174	338.8929	2.220297

Tables 1 and 2; give the comparison of obtained results with the exact solutions [8, 9] for different m, k_1, k_2, k_3 and initial conditions. It can be observed from Tables 1 and 2 that there are an excellent agreement between the results obtained from the MMA and IAFF method and exact one [8, 9]. The maximum relative error between the MMA and IAFF results and exact results is

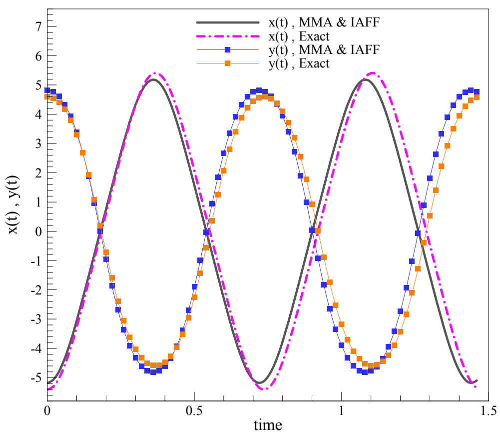

Fig. 3 Comparison of analytical solutions of displacement $x(t)$ and $y(t)$ based on time with the exact solution [9] for $m=10,\ k_1=5,\ k_2=5,\ X_0=10,\ Y_0=20$

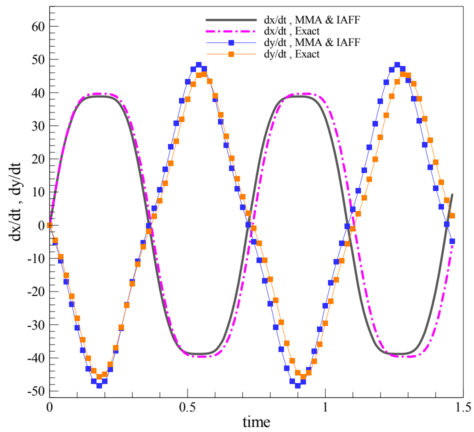

Fig. 4 Comparison of analytical solutions of dx/dt and dy/dt based on time with the exact solution [9] for $m=10,\ k_1=5,\ k_2=5,\ X_0=10,\ Y_0=20$

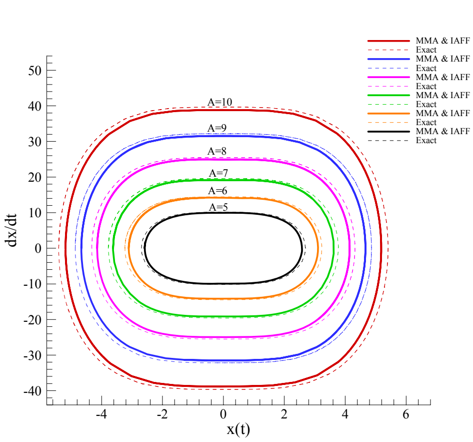

Fig. 5 Comparison of analytical solutions of dx/dt based on $x(t)$ with the exact solution [9] for $m=10$, $k_1=5,\ k_2=5$

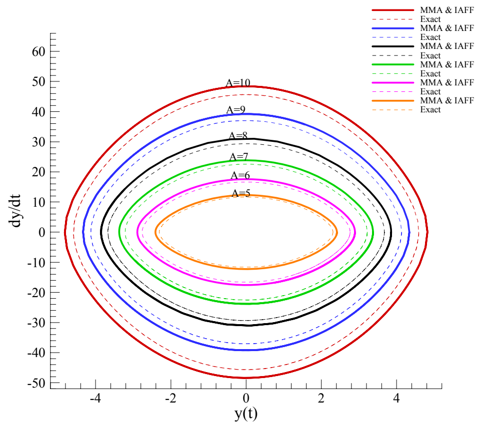

Fig. 6 Comparison of analytical solutions of dy/dt based on $y(t)$ with the exact solution [9] for $m=10$, $k_1=5,\ k_2=5$

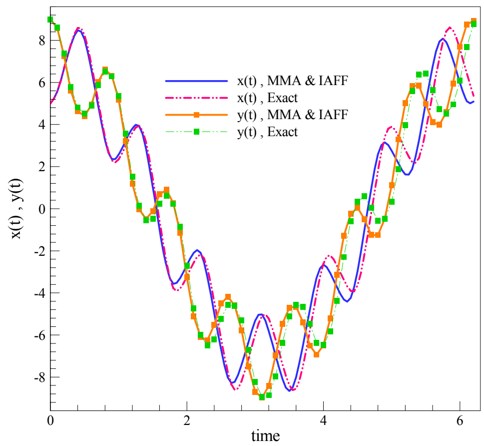

Fig. 7 Comparison of analytical solutions of displacement $x(t)$ and $y(t)$ based on time with the exact solution [8] for $m=1,\ k_1=1,\ k_2=1,\ k_3=2,\ X_0=5,\ Y_0=1$

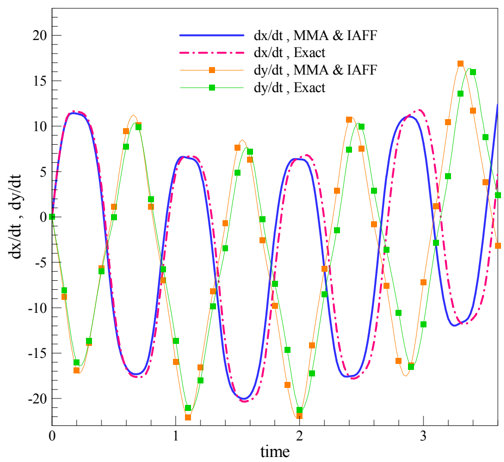

Fig. 8 Comparison of analytical solutions of dx/dt and dy/dt based on time with the exact solution [8] for $m=1,\ k_1=1,\ k_2=1,\ k_3=2,\ X_0=5,\ Y_0=1$

2.220415%.

A comparison of the time history oscillatory displacement response for the two masses with exact solutions are presented in Figs. 3-6 for example 1 and Figs. 7-10 for example 2. From the Figs. 3 and 7, motions

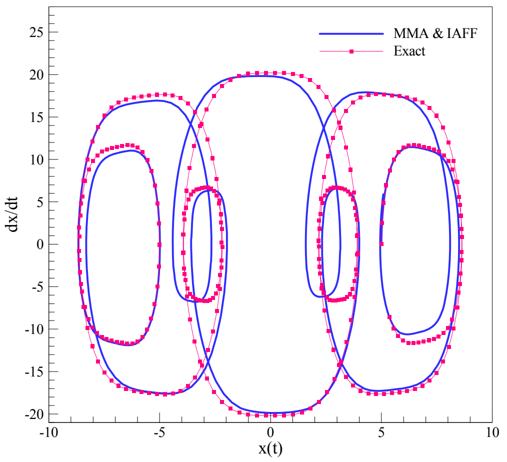

Fig. 9 Comparison of analytical solutions of dx/dt based on $x(t)$ with the exact solution [8] for $m = 1$, $k_1 = 1$, $k_2 = 1$, $k_3 = 2$, $X_0 = 5$, $Y_0 = 1$

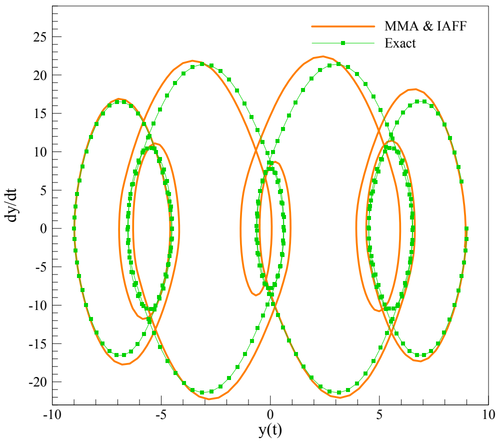

Fig. 10 Comparison of analytical solutions of dy/dt based on $y(t)$ with the exact solution [8] for $m = 1$, $k_1 = 1$, $k_2 = 1$, $k_3 = 2$, $X_0 = 5$, $Y_0 = 1$

of the systems are periodic motions and the amplitude of vibrations is function of the initial conditions. As shown in Figs. 3-10, it is apparent that the MMA and IAFF have an excellent agreement with the numerical solution using the exact solution. These expressions are valid for a wide range of vibration amplitudes and initial conditions. The proposed methods are quickly convergent and can also be readily generalized to two-degree-of-freedom oscillation systems with quadratic nonlinearity by combining the transformation technique. The accuracy of the results shows that the MMA and IAFF can be potentially used for the analysis of strongly nonlinear vibration problems with high accuracy.

6. Conclusion

Two powerful explicit analytical approaches have been developed for a set of second-order coupled differential equations with cubic nonlinearities that govern the nonlinear free vibration of conservative two degree of freedom systems. The solutions have been achieved using the MMA and IAFF. Excellent agreement between approximate frequencies and the exact one are demonstrated and discussed. The methods which are proved to be powerful mathematical tools for studying of nonlinear oscillators. According to the results, the precision and convergence rate of the solutions increase using MMA and IAFF. In conclusion, two practical examples of two-mass systems with free and fixed ends and with linear and nonlinear stiffnesses have been presented and discussed. The first-order approximate solutions are of a high accuracy. Of course, the accuracy can be improved upon using a higher order approximate solution. The result shows that the proposed method for solving TDOF system problems gives results that are highly consistent with published data and exact solutions. The MMA and IAFFare two well-established methods for the analysis of nonlinear systems and could be easily extended to any nonlinear equations. The achieved results indicated that MMA and IAFF are extremely simple, easy, powerful, and triggers good accuracy.

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M. Bayat, I. Pakar, M. Shahidi

SUJUNGTŲ SISTEMŲ SU KUBINIU NETIESIŠKUMU NETIESINIŲ SVYRAVIMŲ ANALIZĖ

R e z i u m ė

Šiame darbe konservatyvių, sujungtų daugiaspyruoklių sistemų su kubiniu netiesiškumu netiesinių laisvųjų svyravimų analitiniams sprendimams nustatymui gauti taikomi du analitiniai maksimalaus ir minimalaus priartėjimo ir patikslintos amplitudės formavimo metodai. Sprendžiant svarbiausias netiesines diferencialines lygtis, dviejų masių sistemos poslinkius galima nustatyti tiesiogiai naudojant pasiūlytų dviejų metodų tiesines antros eilės diferencialines lygtis, kas ir yra pagrindinis šio tyrimo tikslas. Palyginti su tobulais sprendimais, pirmu svyravimų dažnio priartėjimu gaunama didžiausia abiem metodams toleruotina paklaida 2,220179 %. Tai rodo, kad taikant maksimalaus ir minimalaus priartėjimo ir patikslintos amplitudės ir dažnio formulavimo metodus, netgi vienu priartėjimu pasiekiamas didelis sprendimo tikslumas, kuris galioja plačiame svyravimo amplitudžių diapazone, ką rodo ir pateikti pavyzdžiai.

M. Bayat, I. Pakar, M. Shahidi

ANALYSIS OF NONLINEAR VIBRATION OF COUPLED SYSTEMS WITH CUBIC NONLINEARITY

S u m m a r y

In this paper, two powerful analytical methods, called He's Max-Min Approach (MMA) and He's Improved Amplitude-Formulation (IAFF) are used to obtain the analytical solutions for nonlinear free vibration of a conservative, coupled system of mass-spring system with cubic nonlinearity. Solving the governing nonlinear differential equation where the displacement of the two-mass system can be obtained directly from the linear second-order differential equation using a first-order of those approaches is the main objective of the present study. Comparing with exact solutions, the first approximation to the frequency of oscillation produces tolerable error 2.220179%as the maximum for both approaches. It has indicated that by utilizing the He's Max-Min Approach and He's Improved Amplitude-Frequency Formulation, just one iteration leads us to high accuracy of solutions which are valid for a wide range of vibration amplitudes as indicated in the following examples.

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