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# Determination of displacements and stresses in pressurized thick cylindrical shells with variable thickness using perturbation technique

**Mehdi Ghannad*, Gholam Hosein Rahimi**, Mohammad Zamani Nejad*****
**Mechanical Engineering Faculty, Shahrood University of Technology, Shahrood, Iran,*
*E-mail: ghannad.mehdi@gmail.com*
***Mechanical Engineering Department, Tarbiat Modares University, Tehran, Iran, E-mail: rahimi_gh@modares.ac.ir*
****Mechanical Engineering Department, Yasouj University, Yasouj P. O. Box: 75914-353 Iran,*
*E-mail: m.zamani.n@gmail.com, m_zamani@mail.yu.ac.ir*
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## 1. Introduction

Shells are common structural elements in many engineering applications, including pressure vessels, submarine hulls, ship hulls, wings and fuselages of airplanes, containment structures of nuclear power plants, pipes, exteriors of rockets, missiles, automobile tires, concrete roofs, chimneys, cooling towers, liquid storage tanks, and many other structures [1]. They are also found in nature in the form of eggs, leaves, inner ear, skulls, and geological formations [1]. Given the limitations of the classic theories of thick wall shells, very little attention has been paid to the analytical solution of these shells.

Assuming the transverse shear effect, Naghdi and Cooper [2], formulated the theory of shear deformation. The solution of thick cylindrical shells of homogenous and isotropic materials, using the first-order shear deformation theory (FSDT) derived by Mirsky and Hermann [3]. Greenspon [4], opted to make a comparison between the findings regarding the different solutions obtained for cylindrical shells. A paper was also published by Kang and Leissa [5] where equations of motion and energy functionals were derived for a three-dimensional coordinate system. The field equations are utilized to express them in terms of displacement components. Assuming that a heterogeneous system is composed of the elements with different properties, in the paper [6] the reactions of pipeline systems to shock impact load and the possibilities of the simulation and evaluation of dynamic processes are investigated. The layers are made of isotropic, homogeneous, linearly elastic material, and they are considered as concentric cylinders. A complete and consistent 3D set of field equations has been developed by tensor analysis to characterize the behavior of FGM (functionally graded material) thick shells of revolution with arbitrary curvature and variable thickness along the meridional direction [7].

Ghannad et al. [8], making use of the FSDT obtained analytical solution for homogeneous and isotropic truncated thick conical shell. Ghannad and Zamani Nejad [9], obtained the differential equations governing the homogenous and isotropic axisymmetric thick-walled cylinders with the same boundary conditions at the two ends were generally derived, making use of FSDT and the virtual work principle. Following that, the set of nonhomogenous linear differential equations for the cylinder with clamped-clamped ends was solved.

In the present study, the general solution of the clamped-clamped thick cylindrical shells with variable thickness subjected to constant internal pressure will be presented, making use of the FSDT. The governing equations, which are a system of nonhomogenous linear differential equations with variable coefficients, have been solved analytically using the matched asymptotic method (MAM) of the perturbation techniques.

## 2. Analysis

In the FSDT, the sections that are straight and perpendicular to the mid-plane remain straight but not necessarily perpendicular after deformation and loading. In this case, shear strain and shear stress are taken into consideration. In the classical theory of shells, the assumption is that the sections that are straight and perpendicular to the mid-plane remain in the same position even after deformation.

Geometry of the cylinder with variable thickness is shown in Fig. 1. The location of a typical point $m$, $r$ within the shell element may be determined by $R$ and $z$ as

$$r = R\left(x\right) + z \tag{1}$$

where $R$ represents the distance of middle surface from the axial direction, and $z$ is the distance of typical point from the middle surface.

In Eq. (1) $x$ and $z$ must be within the following ranges

$$0 \le x \le L,\ -h/2 \le z \le h/2 \tag{2}$$

where $h$ and $L$ are the thickness and the length of the cylinder.

$R(x)$ and inner and outer radii $\left(r_i,\ r_o(x)\right)$ of the cylinder are as follows

$$\left.\begin{aligned} R(x) &= \frac{r_i + r_o(x)}{2},\ h(x) = r_o(x) - r \\ r_i &= R(x) - \frac{h(x)}{2} = const.,\ r_o(x) = R(x) + \frac{h(x)}{2} \end{aligned}\right\} \tag{3}$$

The general axisymmetric displacement field $\left(U_x, U_z\right)$, in the first-order Mirsky-Hermann's theory could be expressed on the basis of axial displacement and radial displacement, as follows

$$\left.\begin{aligned} U_x(x,z) &= u(x) + \phi(x)z \\ U_\theta &= 0 \\ U_z(x,z) &= w(x) + \psi(x)z \end{aligned}\right\} \tag{4}$$

where $u(x)$ and $w(x)$ are the displacement components of the middle surface. Also, $\phi(x)$ and $\psi(x)$ are the functions used to determine the displacement field.

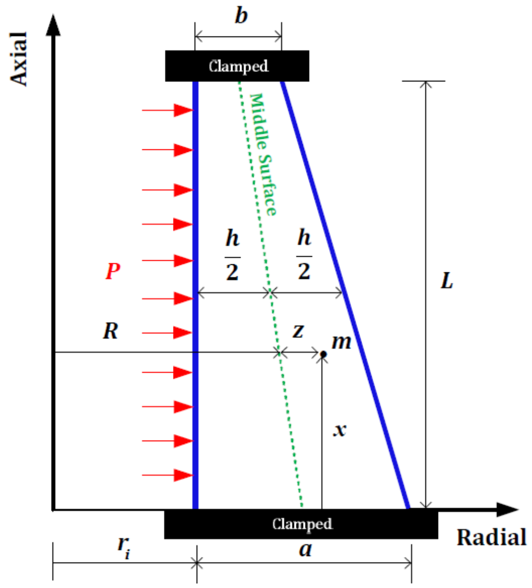


Fig. 1 Geometry of the clamped-clamped cylinder with variable thickness

The strain-displacement relations in the cylindrical coordinates system are

$$\left.\begin{aligned}
\varepsilon_x &= \frac{\partial U_x(x,z)}{\partial x} = \frac{du(x)}{dx} + \frac{d\phi(x)}{dx}z \\
\varepsilon_\theta &= \frac{U_z(x,z)}{r} = \frac{w(x)}{R+z} + \frac{\psi(x)}{R+z}z \\
\varepsilon_z &= \frac{\partial U_z(x,z)}{\partial z} = \psi(x) \\
\gamma_{xz} &= \frac{\partial U_x(x,z)}{\partial z} + \frac{\partial U_z(x,z)}{\partial x} = \\
&= \left(\phi(x) + \frac{dw(x)}{dx}\right) + \frac{d\psi(x)}{dx}z
\end{aligned}\right\} \tag{5}$$

In addition, the stresses on the basis of constitutive equations for homogenous and isotropic materials are as follows

$$\left.\begin{aligned}
\begin{Bmatrix}\sigma_x\\ \sigma_\theta\\ \sigma_z\end{Bmatrix} &= \lambda E \begin{bmatrix}1-\upsilon & \upsilon & \upsilon\\ \upsilon & 1-\upsilon & \upsilon\\ \upsilon & \upsilon & 1-\upsilon\end{bmatrix} \begin{Bmatrix}\varepsilon_x\\ \varepsilon_\theta\\ \varepsilon_z\end{Bmatrix}\\
\tau_{xz} &= \frac{1-2\upsilon}{2}\lambda E \gamma_{xz}
\end{aligned}\right\} \tag{6}$$

where

$$\lambda = \frac{1}{(1+\upsilon)(1-2\upsilon)} \tag{7}$$

$\sigma_i$ and $\varepsilon_i$ are the stresses and strains in the radial $(z)$, circumferential $(\theta)$, and axial $(x)$ directions. $\upsilon$ and $E$ are Poisson's ratio and Young's modulus, respectively.

The normal forces $(N_x, N_\theta, N_z)$, bending moments $(M_x, M_\theta)$, shear force $(Q_x)$, and the twisting moment $(M_{xz})$ in terms of stress resultants are

$$\begin{Bmatrix}N_x\\ N_\theta\\ N_z\end{Bmatrix} = \int_{-h/2}^{h/2} \begin{Bmatrix}\sigma_x\left(1+\frac{z}{R}\right)\\ \sigma_\theta\\ \sigma_z\left(1+\frac{z}{R}\right)\end{Bmatrix} dz \tag{8}$$

$$\begin{Bmatrix}M_x\\ M_\theta\\ M_z\end{Bmatrix} = \int_{-h/2}^{h/2} \begin{Bmatrix}\sigma_x\left(1+\frac{z}{R}\right)\\ \sigma_\theta\\ \sigma_z\left(1+\frac{z}{R}\right)\end{Bmatrix} zdz \tag{9}$$

$$Q_x = \int_{-h/2}^{h/2} \tau_{xz}\left(1+\frac{z}{R}\right)dz \tag{10}$$

$$M_{xz} = \int_{-h/2}^{h/2} \tau_{xz}\left(1+\frac{z}{R}\right)zdz \tag{11}$$

On the basis of the principle of virtual work, the variations of strain energy are equal to the variations of the external work as follows

$$\delta U = \delta W \tag{12}$$

where $U$ is the total strain energy of the elastic body and $W$ is the total external work due to internal pressure. The strain energy is

$$U = \iiint_V U^* dV,\ dV = rdrd\theta dx = (R+z)dxd\theta dz \tag{13}$$

where

$$U^* = \frac{1}{2}\left(\sigma_x\varepsilon_x + \sigma_\theta\varepsilon_\theta + \sigma_z\varepsilon_z + \tau_{xz}\gamma_{xz}\right) \tag{14}$$

and the external work is

$$\left.\begin{aligned}
W &= \iint_S \left(\vec{f}\,\vec{u}\right)dS,\ dS = r_i d\theta dx = \left(R - \frac{h}{2}\right)d\theta dx\\
\vec{f}\,\vec{u} &= PU_z
\end{aligned}\right\} \tag{15}$$

where $P$ is internal pressure.

The variation of the strain energy is

$$\delta U = \int_0^{2\pi}\int_0^L\int_{-h/2}^{h/2} \delta U^* \left(1+\frac{z}{R}\right)Rdzdxd\theta \tag{16}$$

The resulting Eq. (16) will be

$$\begin{aligned}
\frac{\delta U}{2\pi} &= \int_0^L\int_{-h/2}^{h/2} (\sigma_x\delta\varepsilon_x + \sigma_\theta\delta\varepsilon_\theta + \\
&+ \sigma_z\delta\varepsilon_z + \tau_{xz}\delta\gamma_{xz})\left(1+\frac{z}{R}\right)Rdzdx
\end{aligned} \tag{17}$$

and the variation of the external work is

$$\delta W=\int_0^{2\pi}\int_0^{L}\left(P\delta U_z\right)\left(R-\frac{h}{2}\right)dxd\theta \tag{18}$$

The resulting Eq. (18) will be

$$\frac{\delta W}{2\pi}=P\int_0^{L}\delta U_z\left(R-\frac{h}{2}\right)dx \tag{19}$$

Substituting Eqs. (5), (6) and (8) to (11) into Eqs. (17) and (19), and drawing upon calculus of variation and the virtual work principle, we will have

$$\left.\begin{aligned}&\frac{d}{dx}\left(RN_x\right)=0\\&\frac{d}{dx}\left(RM_x\right)-RQ_x=0\\&\frac{d}{dx}\left(RQ_x\right)-N_\theta=-P\left(R-\frac{h}{2}\right)\\&\frac{d}{dx}\left(RM_{xz}\right)-M_\theta-RN_z=P\frac{h}{2}\left(R-\frac{h}{2}\right)\end{aligned}\right\} \tag{20}$$

and the boundary conditions are

$$\left[R\left(N_x\delta u+M_x\delta\phi+Q_x\delta w+M_{xz}\delta\psi\right)\right]_0^L=0 \tag{21}$$

Eq. (21) states the boundary conditions which must exist at the two ends of the cylinder.

We assume that Young's modulus and the Poisson's ratio are constant. Using Eqs. (5) to (11), stress resultants (forces and moments) are obtained in terms of displacements

$$\left.\begin{aligned}&N_x=\lambda Eh\left[(1-\upsilon)\left(\frac{du}{dx}+\frac{h^2}{12R}\frac{d\phi}{dx}\right)+\right.\\&\left.+\frac{\upsilon}{R}w+\upsilon\psi\right]\\&N_\theta=\lambda Eh\left[\upsilon\frac{du}{dx}+\frac{(1-\upsilon)\alpha}{h}w+\right.\\&\left.+\left(1-\frac{(1-\upsilon)R\alpha}{h}\right)\psi\right]\\&N_z=\lambda Eh\left[\upsilon\left(\frac{du}{dx}+\frac{h^2}{12R}\frac{d\phi}{dx}\right)+\right.\\&\left.+\frac{\upsilon}{R}w+(1-\upsilon)\psi\right]\end{aligned}\right\} \tag{22}$$

$$\left.\begin{aligned}&M_x=\lambda E\frac{h^3}{12R}\left[(1-\upsilon)\left(\frac{du}{dx}+R\frac{d\phi}{dx}\right)+2\upsilon\psi\right]\\&M_\theta=\lambda E\frac{h^3}{12R}\left\{\upsilon R\frac{d\phi}{dx}+\frac{12R}{h^3}(1-\upsilon)\left[\left(h-R\alpha\right)w+\right.\right.\\&\left.\left.+\left(R^2\alpha-Rh\right)\psi\right]\right\}\\&M_z=\lambda E\frac{h^3}{12R}\left[\upsilon\frac{du}{dx}+R\frac{d\phi}{dx}+\psi\right]\end{aligned}\right\} \tag{23}$$

$$Q_x=K\left(0.5-\upsilon\right)\lambda Eh\left[\phi+\frac{dw}{dx}+\frac{h^2}{12R}\frac{d\psi}{dx}\right] \tag{24}$$

$$M_{xz}=K\left(0.5-\upsilon\right)\lambda E\frac{h^3}{12R}\left[\phi+\frac{dw}{dx}+R\frac{d\psi}{dx}\right] \tag{25}$$

where $K$ is the shear correction factor and it is assumed that in the static state, for cylindrical shells $K=5/6$ [10]. The parameters $\mu$ and $\alpha$ are as follows

$$\mu=K\left(0.5-\upsilon\right),\ \alpha=ln\left(\frac{2R+h}{2R-h}\right) \tag{26}$$

Substituting Eqs. (22) to (25) in Eqs. (20), a set of nonhomogeneous differential equations with variable coefficients is obtained

$$\left.\begin{aligned}&\frac{d}{dx}\left(\left[B_1\right]\frac{d}{dx}\{y\}\right)+\frac{d}{dx}\left(\left[B_2\right]\{y\}\right)+\\&+\left[B_3\right]\frac{d}{dx}\{y\}+\left[B_4\right]\{y\}=\{F'\}\\&\{y\}=\{u(x)\quad\phi(x)\quad w(x)\quad\psi(x)\}^T\end{aligned}\right\} \tag{27}$$

where $\left[B_1\right]$ to $\left[B_4\right]$ and $\{F'\}$ are as follows

$$\left[B_1\right]=\begin{bmatrix}(1-\upsilon)Rh & (1-\upsilon)\frac{h^3}{12} & 0 & 0\\(1-\upsilon)\frac{h^3}{12} & (1-\upsilon)\frac{Rh^3}{12} & 0 & 0\\0 & 0 & \mu Rh & \mu\frac{h^3}{12}\\0 & 0 & \mu\frac{h^3}{12} & \mu\frac{Rh^3}{12}\end{bmatrix} \tag{28}$$

$$\left[B_2\right]=\begin{bmatrix}0 & 0 & \upsilon h & \upsilon Rh\\0 & 0 & 0 & \upsilon\frac{h^3}{6}\\0 & \mu Rh & 0 & 0\\0 & \mu\frac{h^3}{12} & 0 & 0\end{bmatrix} \tag{29}$$

$$\left[B_3\right]=\begin{bmatrix}0 & 0 & 0 & 0\\0 & 0 & -\mu Rh & -\mu\frac{h^3}{12}\\-\upsilon h & 0 & 0 & 0\\-\upsilon Rh & -\upsilon\frac{h^3}{6} & 0 & 0\end{bmatrix} \tag{30}$$

$$\left[B_4\right]=\begin{bmatrix}0 & 0 & 0 & 0\\0 & -\mu Rh & 0 & 0\\0 & 0 & -(1-\upsilon)\alpha & -h+(1-\upsilon)R\alpha\\0 & 0 & -h+(1-\upsilon)R\alpha & -(1-\upsilon)R^2\alpha\end{bmatrix} \tag{31}$$

$$\{F'\}=\frac{P}{\lambda E}\left(R-\frac{h}{2}\right)\begin{Bmatrix}0\\0\\-1\\\frac{h}{2}\end{Bmatrix} \tag{32}$$

To solve the set of differential equations above, the inverse of the matric $[B_4]$ will be needed. To do this, we take $du/dx$ as $v$, and integrating the first equation in the set of Eqs. (20),

$$RN_x = C_0 \tag{33}$$

Thus, set of differential equations (27) could be derived as follows

$$\left.\begin{array}{l} \frac{d}{dx}\left([A_1]\frac{d}{dx}\{y\}\right)+\frac{d}{dx}\left([A_2]\{y\}\right)+ \\ +[A_3]\frac{d}{dx}\{y\}+[A_4]\{y\}=\{F\} \\ \{y\}=\{v(x)\quad \phi(x)\quad w(x)\quad \psi(x)\}^T \end{array}\right\} \tag{34}$$

where $[A_1]$ to $[A_4]$ and $\{F\}$ are as follows

$$[A_1]=\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & (1-\upsilon)\frac{Rh^3}{12} & 0 & 0 \\ 0 & 0 & \mu Rh & \mu\frac{h^3}{12} \\ 0 & 0 & \mu\frac{h^3}{12} & \mu\frac{Rh^3}{12} \end{bmatrix} \tag{35}$$

$$[A_2]=\begin{bmatrix} 0 & 0 & 0 & 0 \\ (1-\upsilon)\frac{h^3}{12} & 0 & 0 & \upsilon\frac{h^3}{6} \\ 0 & \mu Rh & 0 & 0 \\ 0 & \mu\frac{h^3}{12} & 0 & 0 \end{bmatrix} \tag{36}$$

$$[A_3]=\begin{bmatrix} 0 & (1-\upsilon)\frac{h^3}{12} & 0 & 0 \\ 0 & 0 & -\mu Rh & -\mu\frac{h^3}{12} \\ 0 & 0 & 0 & 0 \\ 0 & -\upsilon\frac{h^3}{6} & 0 & 0 \end{bmatrix} \tag{37}$$

$$[A_4]=\begin{bmatrix} (1-\upsilon)Rh & 0 & \upsilon h & \upsilon Rh \\ 0 & -\mu Rh & 0 & 0 \\ -\upsilon h & 0 & -(1-\upsilon)\alpha & -h+(1-\upsilon)R\alpha \\ -\upsilon Rh & 0 & -h+(1-\upsilon)R\alpha & -(1-\upsilon)R^2\alpha \end{bmatrix} \tag{38}$$

$$\{F\}=\frac{1}{\lambda E}\left\{\begin{array}{c} C_0 \\ 0 \\ -P\left(R-\frac{h}{2}\right) \\ P\frac{h}{2}\left(R-\frac{h}{2}\right) \end{array}\right\} \tag{39}$$

Eqs. (34) is a set of linear non-homogenous differential equations with variable coefficients. For the purpose of solving, MAM of the perturbation theory has been used.

## 3. Perturbation technique

Solving the differential equations with variable coefficients gives rise to solving a system of algebraic equations with variable coefficients and two systems of differential equations with constant coefficients.

These systems of equations have the closed forms solutions. To accomplish this, making use of the characteristic scales, the governing equations are made dimensionless.

$$\left.\begin{array}{lll} x^*=\frac{x}{L} \;, & z^*=\frac{z}{h_0} \;, & h^*=\frac{h}{h_0} \\ R^*=\frac{R}{h_0} \;, & u^*=\frac{u}{h_0} \;, & w^*=\frac{w}{h_0} \end{array}\right\} \tag{40}$$

where $h_0$ is the characteristic thickness. Substituting dimensionless parameters the set of Eqs. (34) is

$$\left.\begin{array}{l} \varepsilon^2\frac{d}{dx^*}\left([A_1^*]\frac{d}{dx^*}\{y^*\}\right)+\varepsilon\left[\frac{d}{dx^*}\left([A_2^*]\{y^*\}\right)+\right. \\ \left.+[A_3^*]\frac{d}{dx^*}\{y^*\}\right]+[A_4^*]\{y^*\}=\{F^*\} \\ \{y^*\}=\{v\quad \phi\quad w^*\quad \psi\}^T \end{array}\right\} \tag{41}$$

where $\varepsilon=\frac{h_0}{L}$ is the perturbation parameter.

$$v=\frac{du}{dx}=\varepsilon\frac{du^*}{dx^*}=\frac{h_0}{L}\frac{du^*}{dx^*} \tag{42}$$

The coefficients matrices $[A_i^*]_{4\times4}$, and force vector $\{F^*\}$ are obtained as follows

$$[A_1^*]=\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{(1-\upsilon)}{12}R^*h^{*3} & 0 & 0 \\ 0 & 0 & \mu R^*h^* & \frac{\mu}{12}h^{*3} \\ 0 & 0 & \frac{\mu}{12}h^{*3} & \frac{\mu}{12}R^*h^{*3} \end{bmatrix} \tag{43}$$

$$\left[A_2^*\right]=\begin{bmatrix} 0 & 0 & 0 & 0 \\ \frac{(1-\upsilon)}{12}h^{*3} & 0 & 0 & \frac{\upsilon}{6}h^{*3} \\ 0 & \mu R^* h^* & 0 & 0 \\ 0 & \frac{\mu}{12}h^{*3} & 0 & 0 \end{bmatrix} \qquad (44)$$

$$\left[A_3^*\right]=\begin{bmatrix} 0 & \frac{(1-\upsilon)}{12}h^{*3} & 0 & 0 \\ 0 & 0 & -\mu R^* h^* & -\frac{\mu}{12}h^{*3} \\ 0 & 0 & 0 & 0 \\ 0 & -\frac{\upsilon}{6}h^{*3} & 0 & 0 \end{bmatrix} \qquad (45)$$

$$\left[A_4^*\right]=\begin{bmatrix} (1-\upsilon)R^*h^* & 0 & \upsilon h^* & \upsilon R^* h^* \\ 0 & -\mu R^* h^* & 0 & 0 \\ -\upsilon h^* & 0 & -(1-\upsilon)\alpha & -h^*+(1-\upsilon)\alpha R^* \\ -\upsilon R^* h^* & 0 & -h^*+(1-\upsilon)\alpha R^* & -(1-\upsilon)\alpha R^{*2} \end{bmatrix} \qquad (46)$$

$$\left\{F^*\right\}=\frac{1}{\lambda E}\begin{Bmatrix} \frac{C_o}{h_0^2} \\ 0 \\ -\frac{P}{2}\left(2R^*-h^*\right) \\ \frac{Ph^*}{4}\left(2R^*-h^*\right) \end{Bmatrix} \qquad (47)$$

where the parameters are as follows

$$\alpha = ln\left(\frac{R^*+\frac{h^*}{2}}{R^*-\frac{h^*}{2}}\right) \qquad (48)$$

The set of Eq. (41) is singular. Therefore, its solution must be considered in the area of boundary layer problems. For the purpose of solving, MAM of the perturbation technique has been used. As boundary conditions are clamped-clamped, one lies in $x^*=0$ and the other in $x^*=1$. So, the solution of the problem contains an outer solution away from the boundaries and two inner solutions near the two boundaries $x^*=0$ and $x^*=1$ [11].

The problem solving is carried out in three areas: 1 - area away from the boundary (outer solution), 2 - boundary area $x=0$ (inner solution at $x^*=0$), 3 - boundary area $x=L$ (inner solution at $x^*=1$). Final solution is obtained by combining the solutions above.

### 3.1. Outer solution

In outer solution, which is carried out at the points away from the boundaries, the solution is assumed as a uniform perturbation series

$$\left\{y_{out}^*\right\}=\sum_{n=0}^{\infty}\varepsilon^n\left\{y_n\left(x^*\right)\right\}=\left\{y_0\right\}+\varepsilon\left\{y_1\right\}+\ldots \qquad (49)$$

With substituting Eq. (49) into Eqs. (41) and putting the same coefficients of zero and first orders based on parameter $\varepsilon$, the following equations are obtained

$$\left.\begin{aligned} &\varepsilon^0:\left[A_4^*\right]\left\{y_0\right\}=\left\{F^*\right\} \\ &\varepsilon^1:\left[A_4^*\right]\left\{y_1\right\}+\frac{d}{dx^*}\left(\left[A_2^*\right]\left\{y_0\right\}\right)+\left[A_3^*\right]\frac{d}{dx^*}\left\{y_0\right\}=0 \end{aligned}\right\} \qquad (50)$$

Eqs. (50) are a system of algebraic equations with variable coefficients. Solving this set of equations using inverse matrix method $\left\{y_0\right\}$ and $\left\{y_1\right\}$ are obtained, respectively. Therefore, the outer solution is as follows

$$\left\{y_{out}^*\right\}=\left\{y_0\right\}+\varepsilon\left\{y_1\right\} \qquad (51)$$

### 3.2. Inner solution

This solution, which is carried out at points near the boundaries, due to fast changes in the boundaries, the fast variable of $\left(x^*-a\right)/\varepsilon$ is used in order to observe the changes in boundary areas. For inner solution, Taylor expansion around point *a* must be given to the coefficients matrices and force vector

$$\left[A_i^*\left(x^*\right)\right]=\left[A_i^*\left(a\right)\right]+\sum_{n=1}^{\infty}\frac{\left(x^*-a\right)^n}{n!}\left[\frac{d^nA_i^*}{dx^{*n}}\right]_{x^*=a} \qquad (52)$$

$$\left\{F^*\left(x^*\right)\right\}=\left\{F^*\left(a\right)\right\}+\sum_{n=1}^{\infty}\frac{\left(x^*-a\right)^n}{n!}\left[\frac{d^nF^*}{dx^{*n}}\right]_{x^*=a} \qquad (53)$$

The solution of the equation is assumed as a uniform perturbation series in terms of the fast variable

$$\left\{y_{in}^*\right\}=\sum_{n=0}^{\infty}\varepsilon^n\left\{Y_n\left(\frac{x^*-a}{\varepsilon}\right)\right\} \qquad (54)$$

Substituting Eq. (53) to (55) into Eqs. (41) and putting the same coefficients of zero and first orders based on parameter $\varepsilon$, the set of linear differential equations with constant coefficients are obtained.

Differential operators are defined as follows

$$\left.\begin{aligned} &L\left(y,x|a\right)=\left[A_1^*\left(a\right)\right]\frac{d^2}{dx^2}\left\{y\right\}+\left(\left[A_2^*\left(a\right)\right]+\left[A_3^*\left(a\right)\right]\right)\frac{d}{dx}\left\{y\right\}+\left[A_4^*\left(a\right)\right]\left\{y\right\} \\ &M\left(y,x|a\right)=\left[\frac{dA_1^*}{dx^*}\right]_{x^*=a}\frac{d}{dx}\left(x\frac{d}{dx}\left\{y\right\}\right)+\left[\frac{dA_2^*}{dx^*}\right]_{x^*=a}\frac{d}{dx}\left(x\left\{y\right\}\right)+\left[\frac{dA_3^*}{dx^*}\right]_{x^*=a}\left(x\frac{d}{dx}\left\{y\right\}\right)+\left[\frac{dA_4^*}{dx^*}\right]_{x^*=a}\left(x\left\{y\right\}\right) \end{aligned}\right\} \qquad (55)$$

In the boundary $x^*=0$, the fast variable is defined as follows

$$\eta=\frac{x-0}{h_0}=\frac{x^*}{\varepsilon} \quad (56)$$

At this point, $x^*$ is substituted into Eqs. (52) to (55), and then the results are substituted into Eqs. (41). Putting the same coefficients of zero and first orders based on parameter $\varepsilon$, the following equations are obtained

$$\left.\begin{aligned}&\varepsilon^0: L\left(V_0,\eta|0\right)=\left\{F^*(0)\right\}\\&\varepsilon^1: L\left(V_1,\eta|0\right)+M\left(V_0,\eta|0\right)=\\&=\eta\left\{\frac{dF^*}{dx^*}\right\}_0\end{aligned}\right\} \quad (57)$$

The set of differential Eq. (57) has both general and particular solutions. The solution of these equations yields $\{V_0\}$ and $\{V_1\}$ [12]

$$\left.\begin{aligned}&\{V_0\}=\{V_0\}_g+\{V_0\}_p\\&\{V_1\}=\{V_1\}_g+\{V_1\}_p\\&\left\{y^*_{in(0)}\right\}=\{Y_0\}=\\&=\{V_0\}+\varepsilon\{V_1\}\end{aligned}\right\} \quad (58)$$

In the boundary $x^*=1$, the fast variable of $\xi$ is defined in the following way

$$\xi=\frac{x-L}{h_0}=\frac{x^*-1}{\varepsilon} \quad (59)$$

At this point, $x^*$ is substituted into Eqs. (52) to (55), and then the results are substituted into Eqs. (41). Putting the same coefficients of zero and first orders based on parameter $\varepsilon$, the following equations are obtained

$$\left.\begin{aligned}&\varepsilon^0: L\left(W_0,\xi|1\right)=\left\{F^*(1)\right\}\\&\varepsilon^1: L\left(W_1,\xi|1\right)+M\left(W_0,\xi|1\right)=\xi\left\{\frac{dF^*}{dx^*}\right\}_1\end{aligned}\right\} \quad (60)$$

The set of differential equations (60) has both general and particular solutions. The solution of these equations yields $\{W_0\}$ and $\{W_1\}$ [12]

$$\left.\begin{aligned}&\{W_0\}=\{W_0\}_g+\{W_0\}_p\\&\{W_1\}=\{W_1\}_g+\{W_1\}_p\\&\left\{y^*_{in(L)}\right\}=\{Y_L\}=\{W_0\}+\varepsilon\{W_1\}\end{aligned}\right\} \quad (61)$$

### 3.3. Composite solution

The composite solution or MAM is the sum of the outer solution and the inner solutions minus the overlapping part. Using MAM, the overlapping part is obtained in the following way [11]: First $\left\{y^*_{in}\right\}$ is written in terms of $x^*$. Following that, the equation obtained is expanded in terms of small values of $\varepsilon$. Now, of the expansion, two terms are considered as the overlapping part. $\{J_0\}$ and $\{J_1\}$ of the overlapping part lie in the area of $x^*=0$ and $x^*=1$, respectively. Finally, using MAM, taken from the perturbation technique, the composite solution, which is an analytical solution, is obtained for the equations governing the cylinder with variable thickness.

$$\begin{aligned}&\left\{y^*\right\}=\left\{y^*_{out}\right\}+\left\{y^*_{in}\right\}-\left\{J_{over}\right\}=\\&=\left\{y^*_{out}\right\}+\left\{Y_0+Y_L\right\}-\left\{J_0+J_L\right\}\end{aligned} \quad (62)$$

## 4. Results and discussion

A cylindrical shell with $r_i=40$ mm, $a=20$ mm, $b=10$ mm, and $L=800$ mm will be considered in this paper. For analytical and numerical results the properties used are $E=200$ GPa and $\upsilon=0.3$. The applied internal pressure is 80 MPa.

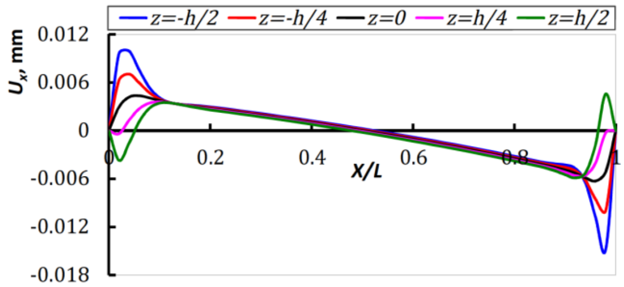


Fig. 2 Axial displacement distribution in different layers

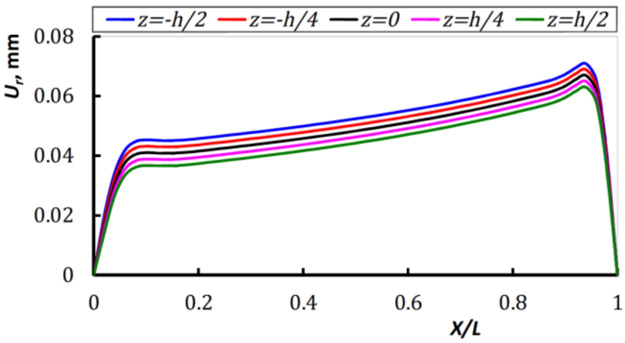


Fig. 3 Radial displacement distribution in different layers

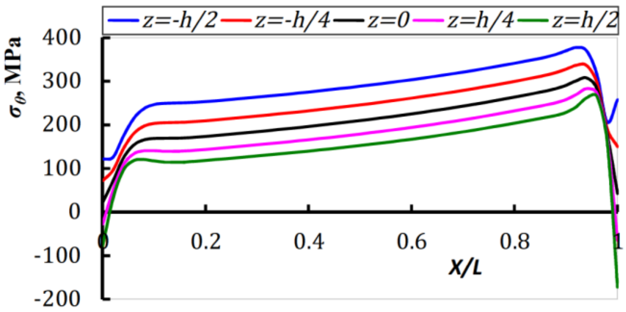


Fig. 4 Circumferential stress distribution in different layers

Fig. 2 shows the distribution of axial displacement at different layers. At points away from the boundaries, axial displacement does not show significant differences in different layers, while at points near the boundaries, the reverse holds true. The distribution of radial displacement at different layers is plotted in Fig. 3. The radial displacement at points away from the boundaries depends on radius and length.

According to Figs. 2 and 3, the change in axial and radial displacements in the upper boundary is greater than that of the lower boundary and the greatest axial and radial displacement occurs in the internal surface $(z=-h/2)$.

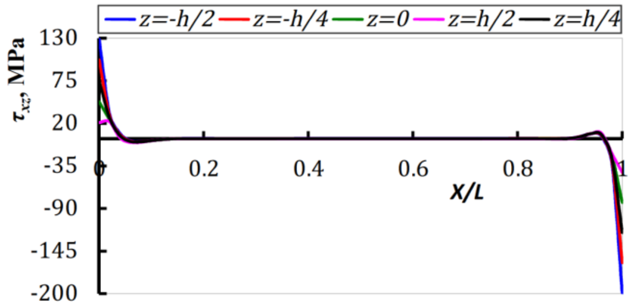


Fig. 5 Shear stress distribution in different layers

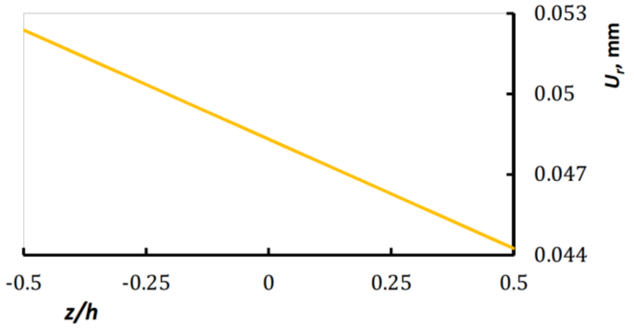


Fig. 6 Radial displacement distribution in $x=L/2$

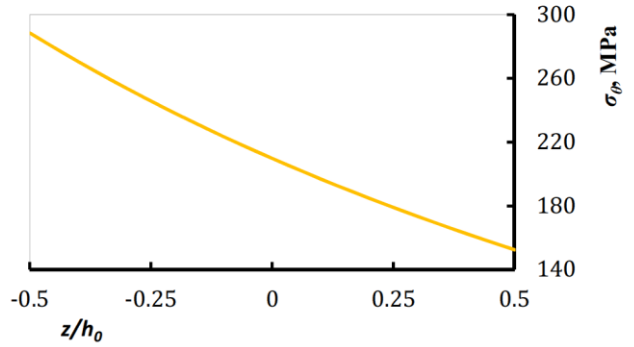


Fig. 7 Circumferential stress distribution in $x=L/2$

Distribution of circumferential stress in different layers is shown in Fig. 4. The circumferential stress at all points depends on radius and length. The greatest circumferential stress occurs in the internal surface $(z=-h/2)$. Fig. 5 shows the distribution of shear stress at different layers. The shear stress at points away from the boundaries at different layers is the same and trivial. However, at points near the boundaries, the stress is significant, especially in the internal surface, which is the greatest. In Figs. 6 and 7, distributions of radial displacement and circumferential stress along radial direction in $x=L/2$ are shown. There is a decrease in the values of the radial displacement and circumferential stress as radius increases.

Displacements and circumferential stress distributions are obtained using FSDT are compared with the solutions of finite element method (FEM) and are presented in the form of graphs in the Figs. 8-10.

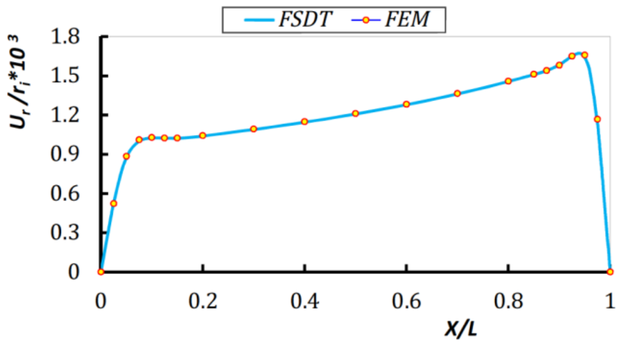


Fig. 8 Radial displacement distribution in middle layer

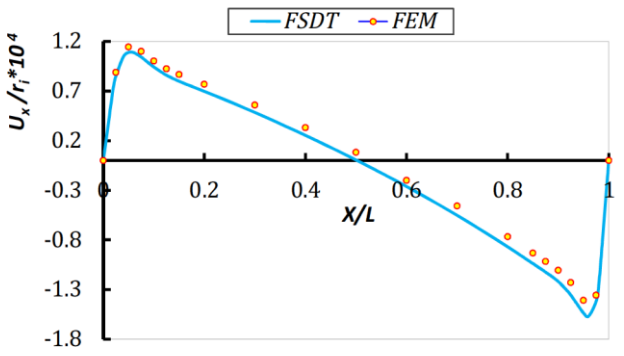


Fig. 9 Axial displacement distribution in middle layer

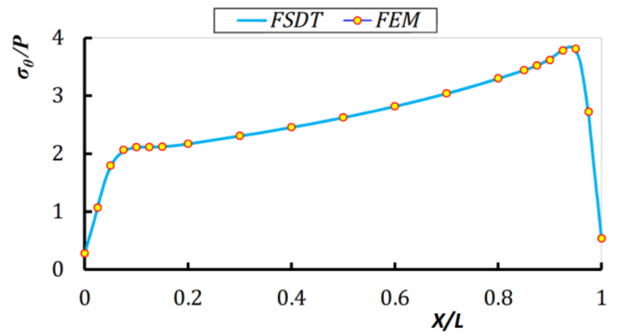


Fig. 10 Circumferential stress distribution in middle layer

## 5. Conclusions

In this study, the analytical solution of a thick homogenous and isotropic cylindrical shell with variable thickness is presented, making use of the FSDT. In line with the energy principle and the FSDT, the equilibrium equations have been derived. Using the MAM of the perturbation theory, the system of differential equations which are ordinary and have variable coefficients has been solved analytically. The axial displacement at points away from the boundaries depends more on the length rather than the radius, whereas at boundaries, this depends on both length and radius. The radial displacement at all points depends on the radius and the length. The circumferential stress at different layers depends on the radius and the length. These changes are relatively great. The greatest values of stress and displacement belong to the inner surface. The shear stress at the points away from the boundaries is insignificant, and at boundary layers it is the opposite.

At the boundary areas, given that displacements and stresses are dependent on radius and length. In the areas further away from the boundaries, as the displacements and stresses along the cylinder remain constant and dependent on radius. The shear stress in boundary areas cannot be ignored, but in areas further away from the boundaries, it can be ignored. The maximum displacements and stresses in all the areas of the cylinder occur on the internal surface. Good agreement was found between the analytical solutions and the solutions carried out through the FEM.

**References**


1. **Reddy, J.N.** 2007. Theory and Analysis of Elastic Plates and Shells, Boca Raton: CRC Press, 403p.
2. **Naghdi, P.M.; Cooper, R.M.** 1956. Propagation of elastic waves in cylindrical shells, including the effects of transverse shear and rotary inertia, J. of the Acoustical Society of America, v.29, No1: 56-63. http://dx.doi.org/10.1121/1.1908222
3. **Mirsky, I.; Hermann, G.** 1958. Axially motions of thick cylindrical shells, J. of Applied Mechanics-Transactions of the ASME, v.25: 97-102.
4. **Greenspon, J.E.** 1960. Vibration of a thick-walled cylindrical shell, comparison of the exact theory with approximate theories, J. of the Acoustical Society of America, v.32, No.5: 571-578. http://dx.doi.org/10.1121/1.1908148
5. **Kang, J.H.; Leissa, A.W.** 2001. Three-dimensional field equations of motion and energy functionals for thick shells of revolution with arbitrary curvature and variable thickness, J. of Applied Mechanics-Transactions of the ASME, v.68, No.6: 953-954. http://dx.doi.org/10.1115/1.1406961
6. **Dorosevas, V.; Volkovas, V.** 2007. Analysis the interaction of two cylindrical surfaces under shock impact load, Mechanika 6(68): 49-52.
7. **Nejad, M.Z.; Rahimi, G.H.; Ghannad, M.** 2009. Set of field equations for thick shell of revolution made of functionally graded materials in curvilinear coordinate system, Mechanika 3(77): 18-26.
8. **Ghannad, M.; Nejad, M.Z.; Rahimi, G.H.** 2009. Elastic solution of axisymmetric thick truncated conical shells based on first-order shear deformation theory, Mechanika 5(79): 13-20.
9. **Ghannad, M.; Nejad, M.Z.** 2010. Elastic analysis of pressurized thick hollow cylindrical shells with clamped-clamped ends, Mechanika 5(85): 11-18.
10. **Vlachoutsis, S.** 1992. Shear correction factors for plates and shells, Int. J. for Numerical Method in Engineering, v.33, No7: 1537-1552. http://dx.doi.org/10.1002/nme.1620330712
11. **Nayfeh, A.H.** 1993. Introduction to perturbation techniques, John Wiley, New York, 265p.
12. **Wylie, C.R.; Barrett, L.C.** 1995. Advanced Engineering Mathematics. McGraw-Hill, 306p.


Mehdi Ghannad, Gholam Hosei Rahimi,
Mohammad Zamani Nejad

HERMETINIŲ STORASIENIŲ KINTAMO STORIO KEVALŲ POSLINKIŲ IR ĮTEMPIŲ NUSTATYMAS NAUDOJANT ŽADINIMO TECHNIKĄ

R e z i u m ė

Straipsnyje pateikiamas storasienio cilindrinio kevalo su standžiai įtvirtintais galais ir kintamo storio sienele, apkrauto pastoviu vidiniu slėgiu, glaustas analitinis sprendimas. Kadangi problemos negalima išspręsti naudojantis plokštumos tamprumo teorija, siūloma remtis pirmos eilės šlyties deformacijos teorija. Remiantis šia teorija ir virtualaus darbo principu sudarytos ašiai simetrinių storasienių cilindrinių kevalų su kintamo storio sienele deformacijos lygtys. Pagrindinės lygybės – tai paprastųjų diferencialinių lygčių sistema su kintamais koeficientais. Naudojant su žadinimo technika suderintą asimptominį metodą, šios lygybės gali būti keičiamos į algebrinių lygčių sistemą su kintamais koeficientais ir dvi diferencialinių lygčių sistemas su pastoviais koeficientais.

Mehdi Ghannad, Gholam Hosein Rahimi,
Mohammad Zamani Nejad

DETERMINATION OF DISPLACEMENTS AND STRESSES IN PRESSURIZED THICK CYLINDRICAL SHELLS WITH VARIABLE THICKNESS USING PERTURBATION TECHNIQUE

S u m m a r y

This article presents a closed form analytical solution for clamped-clamped thick cylindrical shells with variable thickness subjected to constant internal pressure. Regarding the problem which could not be solved through plane elasticity theory (PET), the solution based on the first-order shear deformation theory (FSDT) is suggested. Based FSDT on, and the virtual work principle, the differential equations governing axisymmetric thick cylindrical shells with variable thickness have been derived. The governing equations are a system of ordinary differential equations with variable coefficients. Using the matched asymptotic method (MAM) of the perturbation technique, these equations could be converted into a system of algebraic equations with variable coefficients and two systems of differential equations with constant coefficients.

**Keywords:** displacements, stresses, pressurized thick cylindrical shells, perturbation technique.

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