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Analytical solutions of the tensile strength and preponderant crack angle for the I-II mixed crack in brittle material

Zhang Yan*, Ji Hongguang, Ye Jianhong***, Li Shiyu******

**Department of Civil and Environmental Engineering, University of Science and Technology Beijing, Beijing, 100083 China, E-mail: yanzhang2009@gmail.com*

***Department of Civil and Environmental Engineering, University of Science and Technology Beijing, Beijing, 100083 China, E-mail: jihongguang@ces.ustb.edu.cn*

****Division of Civil Engineering, University of Dundee, Dundee, DD1 4HN UK, E-mail: jzye@dundee.ac.uk*

****Key Lab. of Eng. & Geomech., Institute of Geology and Geophysics, Chinese Academy of Sci., Beijing, 100029 China, E-mail: yejianhongcas@gmail.com*

*****Institute of Geophysics, China Earthquake Administrate, Beijing, 100081, China*

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1. Introduction

Generally, the cracks in real structural materials are loaded by the complicated combined stress field due to the asymmetry of structures and loads, anisotropy of materials or other reasons. It results in that the stress fields around the cracks tip are significantly different from that of pure mode I, II, III cracks. The stress field should be affected by the pure mode I and II, even III cracks simultaneously. The cracks different from pure mode I, II and III cracks are referred to as mixed cracks. In the real structural materials, the mixed cracks exist abundantly, and the I-II mixed cracks is one of the most common forms. The mechanical behaviours of the I-II mixed cracks are always paid a great number of attentions by engineers [1-3].

In the calculation analysis of fracture mechanics, linear-elastic model is a common one and used widely. The linear-elastic model is not only simple and easy to be applied, but also could avoid the problem to some extent that the development of elastoplastic model is not perfect and mature. For those structure materials with large brittleness, the plastic deformation is very small when the brittle fracture occurring under tensile stress loading. Therefore, the linear elastic model is applicable for most brittle materials in solving fracture mechanical problems.

Based on the linear elastic constitutive model, a series of fracture criterions are proposed by some researchers which are applicable for the mixed cracks, such as, the maximum circumferential stress criterion proposed by Erdogan and Sih [4]; the maximum energy release rate criterion developed by Hussain et al. [5]. The maximum tensile strain criterion [6]; the maximum strain energy density factor criterion proposed by Sih [7]; equivalent stress intensity factors criterion [8]; expansion/torsion strain energy density factor criterion [9, 10]; J-integration criterion [11]. Among these fracture criterions, the maximum circumferential stress criterion is the simplest one with excellent applicability, which is frequently used by researchers and engineers [12, 13]. It is indicated by the mixed cracks test using the concretes that fracture angle determined by the maximum circumferential stress criterion agrees well with experimental data [3]. It is shown that the applicability of the maximum circumferential stress criterion is very well for the brittle materials.

In practical engineering, it is important for us to know the critical limited load and the initial fracture angle for propagating once the stress intensity factors K_I, K_{II} of structural materials and fracture toughness K_{IC} of model I crack are calibrated through experimental methods. Recently, a great number of attentions have been pain to investigate the ultimate strength of material with crack [14 - 16]. In this study, the tensile strength and the corresponding preponderant fracture angle for a I-II mixed crack contained in infinite plate under uniaxial tensile stress are investigated based on the linear-elastic maximum circumferential stress criterion. Namely, it will be demonstrated that under how much the tensile stress applied on the infinite plate making the I-II mixed crack begin to propagate, and what the preponderant crack angle is making the I-II mixed crack most easily to propagate in the infinite plate under tensile stress.

2. Maximum circumferential stress criterion

Maximum circumferential stress criterion is proposed by Erdogan and Sih [4] based on the experimental results that the mixed cracks propagate along the direction which is perpendicular with the maximum circumferential tensile stress. The basic statement is:

- cracks unstably propagate along the direction perpendicular with the maximum circumferential stress $\sigma_{\theta\theta}$ near the tips of mixed cracks;

- the condition for cracks beginning to unstably propagate is that the maximum $\sigma_{\theta\theta}$ reaches a certain critical value of the materials (tensile strength).

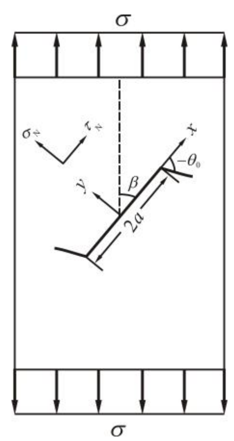

Fig. 1 Force applying sketch of engineering material contain I-II mixed fracture

In the two dimensional model shown in Fig. 1 ($K_{III} = 0$), the stress fields near the tips of the I-II mixed cracks are expressed as (in the form of polar coordinate; and the endpoints of crack is the origin) [17].

$$\left.\begin{aligned} \sigma_{rr} &= \frac{1}{2\sqrt{2\pi r}}\left[K_{\mathrm{I}}\left(3-cos\theta\right)cos\left(\frac{\theta}{2}\right)+K_{\mathrm{II}}\left(3cos\theta-1\right)sin\frac{\theta}{2}\right] \\ \sigma_{\theta\theta} &= \frac{1}{2\sqrt{2\pi r}}cos\frac{\theta}{2}\left[K_{\mathrm{I}}\left(1+cos\theta\right)-3K_{\mathrm{II}}sin\theta\right] \\ \tau_{r\theta} &= \frac{1}{2\sqrt{2\pi r}}cos\frac{\theta}{2}\left[K_{\mathrm{I}}sin\theta+K_{\mathrm{II}}\left(3cos\theta-1\right)\right] \end{aligned}\right\}+o\left(r^{-1/2}\right) \tag{1}$$

where K_{I}, K_{II} are the stress intensity factors of pure mode I and mode II cracks; θ ($\theta[-\pi, \pi]$) is positive for counter-clockwise situations. Otherwise, it is negative. $o(r^{-1/2})$ is the high order small value in Eq. (1), which is ignored in the following derivation. Additionally, the tensile stress is taken as positive value in this study.

2.1. Direction of crack propagating

According to the maximum circumferential stress criterion, cracks should propagate along the direction perpendicular to the maximum $\sigma_{\theta\theta}$ near the tips of I-II mixed crack. The following conditions have to be satisfied

$$\partial\sigma_{\theta\theta}/\partial\theta=0 \quad \text{and} \quad \partial^2\sigma_{\theta\theta}/\partial\theta^2=0 \tag{2}$$

Differentiating at both sides of Eq. (1), we can obtain

$$\frac{\partial\sigma_{\theta\theta}}{\partial\theta}=-\frac{1}{\sqrt{2\pi r}}\frac{3}{2}cos\frac{\theta}{2}\left[K_{\mathrm{I}}sin\theta+K_{\mathrm{II}}(3cos\theta-1)\right] \tag{3}$$

$$\frac{\partial^2\sigma_{\theta\theta}}{\partial\theta^2}=\frac{3}{4\sqrt{2\pi r}}\left\{\begin{aligned} &\frac{1}{2}sin\frac{\theta}{2}\left[K_{\mathrm{I}}sin\theta+K_{\mathrm{II}}\left(3cos\theta-1\right)\right]- \\ &-cos\frac{\theta}{2}\left(K_{\mathrm{I}}cos\theta-3K_{\mathrm{II}}sin\theta\right) \end{aligned}\right\} \tag{4}$$

From $\partial\sigma_{\theta\theta}/\partial\theta=0$, we get

$$cos\frac{\theta}{2}\left[K_{\mathrm{I}}sin\theta+K_{\mathrm{II}}\left(3cos\theta-1\right)\right]=0 \tag{5}$$

A solution of Eq. (5) is that $cos\left(\theta/2\right)=0$ $\left(\theta=\pm\pi, \sigma_{\theta\theta}=0\right)$. However, if substituting them into Eq. (4), it is found that $\partial\sigma_{\theta\theta}/\partial\theta=0$. It can not meet the condition of $\partial^2\sigma_{\theta\theta}/\partial\theta^2<0$. In addition, the fracture surface described by this solution is the same with the surface of mixed cracks. Actually, there is no physical meaning. Finally, the initial fracture angle θ_0 is determined by the following equation

$$K_{\mathrm{I}}sin\theta+K_{\mathrm{II}}\left(3cos\theta-1\right)=0 \tag{6}$$

When both K_{I} and K_{II} are not equal to 0 in Eq. (6), we obtain

$$\theta_0=2arc\,tan\frac{\left[\left(1+\sqrt{1+8\left(K_{\mathrm{II}}/K_{\mathrm{I}}\right)^2}\right)\right]}{4\left(K_{\mathrm{II}}/K_{\mathrm{I}}\right)}$$

Substituting $\theta_0=2arc\,tan\frac{\left[\left(1+\sqrt{1+8\left(K_{\mathrm{II}}/K_{\mathrm{I}}\right)^2}\right)\right]}{4\left(K_{\mathrm{II}}/K_{\mathrm{I}}\right)}$ into Eq. (4), we can obtain $\frac{\partial^2\sigma_{\theta\theta}}{\partial\theta^2}>0$. It means that the $\sigma_{\theta\theta}$ reaches its minimum value, namely the maximum compressive stress. Therefore, the initial fracture angle θ_0 can be determined only by the following equation

$$\theta_0=2arc\,tan\frac{\left[\left(1+\sqrt{1+8\left(K_{\mathrm{II}}/K_{\mathrm{I}}\right)^2}\right)\right]}{4\left(K_{\mathrm{II}}/K_{\mathrm{I}}\right)} \tag{7}$$

The variable curve of θ_0 is shown in Fig. 2.

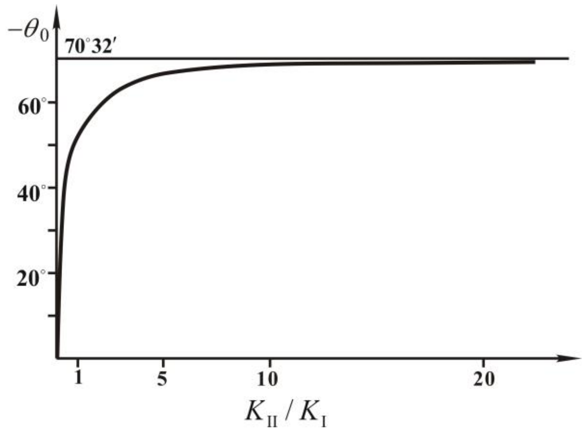

Fig. 2 The relationship between the initial fracture angle θ_0 and the ratio of $K_{\mathrm{II}}/K_{\mathrm{I}}$

From Eq. (6), we know that

$$K_{\mathrm{II}}/K_{\mathrm{I}}=-\frac{sin\theta_0}{3cos\theta_0-1} \tag{8}$$

From Eq. (8), it is found that $3cos\theta_0 - 1 > 0$ and $\theta_0 < 70°32'$ due to that $K_{\mathrm{II}}/K_{\mathrm{I}} > 0$ and $sin\theta_0 < 0$. Therefore, substituting Eq. (6) or Eq. (8) into Eq. (4), we obtain

$$\left.\frac{\partial^2\sigma_{\theta\theta}}{\partial\theta^2}\right|_{\theta=\theta_0}=-\frac{3K_{\mathrm{I}}}{4\sqrt{2\pi r}}cos\frac{\theta_0}{2}\frac{\left(3-cos\theta_0\right)}{\left(3cos\theta_0-1\right)}<0 \tag{9}$$

It is indicated that the solution of Eq. (6) $\theta = \theta_0$ meets the condition letting the $\sigma_{\theta\theta}$ reaches its maximum value.

Comparing Eq. (6) and the third expression in Eq. (1), it is found that $\tau_{r\theta} = 0$ on the plane where the $\sigma_{\theta\theta}$ reaches its maximum value. That is to say the plane is the principle stress plane. Therefore, the maximum circumferential stress is the maximum tensile stress near the tips of the mixed cracks if only the singular term is retained in the all stress components.

In Eq. (6), when $K_{\mathrm{I}} = 0$, $K_{\mathrm{II}} \neq 0$ (pure model II crack), namely, $K_{\mathrm{II}} / K_{\mathrm{I}} \to \infty$, it is derived that

$$\theta_0 = arccos\frac{1}{3} = -70^{\circ}32'. \tag{10}$$

2.2. Stress condition for cracks beginning to propagate

According to maximum circumferential stress criterion, the cracks begin to unstably propagate when $\sigma_{\theta\theta max}$ reaches a certain critical value $\sigma_{\theta\theta c}$ (tensile strength). Generally, this critical value is determined through some experimental methods of pure mode I crack. For a pure mode I cracks, $K_{\mathrm{II}} = 0$. From Eq. (7), we know that the initial facture angle θ_0 equals to 0 for a pure mode I crack which is known as self-similar propagation. When the pure mode I crack begins to propagate, the $\sigma_{\theta\theta max}$ exactly reaches the critical value $\sigma_{\theta\theta c}$ of materials. From this point of view and Eq. (1), it is obtained that

$$\sigma_{\theta\theta c} = \frac{1}{\sqrt{2\pi r}} K_{\mathrm{IC}} \tag{11}$$

where K_{IC} is the fracture toughness of pure model I crack. Once this critical value of materials is determined, and combined with Eq. (1), we know that the instability criterion of I-II mixed cracks is

$$\frac{1}{2}cos\frac{\theta_0}{2}\left[K_{\mathrm{I}}\left(1+cos\theta_0\right)-3K_{\mathrm{II}}sin\theta_0\right]=K_{\mathrm{IC}} \tag{12}$$

In a sense, the I-II mixed cracks can be considered as a kind of equivalent mode I cracks. The equivalent stress intensity factor could be formulated as

$$K_{eff} = \frac{1}{2}cos\frac{\theta_0}{2}\left[K_{\mathrm{I}}\left(1+cos\theta_0\right)-3K_{\mathrm{II}}sin\theta_0\right]$$

Then the instability criterion is

$$K_{eff} = K_{\mathrm{IC}}$$

For pure mode II cracks, the initial fracture angle when the cracks begin to propagate is $\theta_0 = cos^{-1}(1/3)$ which is determined by Eq. (7). At this moment, $K_{\mathrm{II}} = K_{\mathrm{IIC}}$. If substituted into Eq. (12), it is obtained

$$\frac{K_{\mathrm{IIC}}}{K_{\mathrm{IC}}} = \frac{\sqrt{3}}{2}. \tag{13}$$

3. Uniaxial brittle tensile strength and the corresponding crack angle

As it is shown in Fig. 1 there is an inclined crack in infinite plate. In this infinite plate, the far-field stresses can be determined as following expressions according to the coordinate transformation

$$\left.\begin{aligned} \sigma_{yy} &= \sigma sin^2\beta \\ \sigma_{xx} &= \sigma cos^2\beta \\ \tau_{xy} &= \sigma\ sin\beta\ cos\beta \end{aligned}\right\} \tag{14}$$

where β is the sharp angle between the crack and the tensile stress. From the definition of stress intensity factor, we know that

$$\left.\begin{aligned} K_{\mathrm{I}} &= \sigma\sqrt{\pi a}\, sin^2\beta \\ K_{\mathrm{II}} &= \sigma\sqrt{\pi a}\, sin\beta\, cos\beta \end{aligned}\right\} \tag{15}$$

Substituting Eq. (15) into (6), the relationship between the fracture angle θ_0 and the crack angle β could be written as follow

$$tan\beta = \frac{1-3cos\theta_0}{sin\theta_0} \tag{16}$$

From the viewpoint of causal relationship, the crack angle β should be a independent variable, and the fracture angle θ_0 should be the dependent variable. Therefore, equation (16) is an implicit function between β and θ_0. The explicit function between β and θ_0 can be obtained based on the Eqs. (7) and (15)

$$\theta_0 = 2arc\,tan\frac{\left[\left(1-\sqrt{1+8cot^2\beta}\right)\right]}{4cot\beta} \tag{17}$$

Similar with the definition of fracture toughness, we define the tensile brittle capacity of materials as

$$K_J = \sigma_c\sqrt{\pi a} \tag{18}$$

where the σ_c is the far-field tensile stress when the I-II mixed crack begin to propagate, K_J is the equivalent fracture toughness of the I-II mixed crack. When the crack begins to propagate, the far-field tensile stress σ applied to infinite plate reaches σ_c (Fig. 1), and Eqs. (15) and (12) are satisfied simultaneously. Combining Eqs. (15) and (12), and letting $\sigma = \sigma_c$, we obtain

$$K_J = \frac{K_{\mathrm{IC}}}{\frac{1}{4}cos\frac{\theta_0}{2}\left[\left(1+cos\theta_0\right)\left(1-cos2\beta\right)-3sin\theta_0 sin2\beta\right]} \tag{19}$$

Rewriting the above Eq. (19) in another form

$$K_J = \frac{K_{\mathrm{IC}}}{F\left(\theta_0,\beta\right)} \tag{20}$$

in which

$$F\left(\theta,\beta\right) = \frac{1}{4}cos\frac{\theta}{2}\times$$
$$\times\left[\left(1+cos\theta\right)\left(1-cos2\beta\right)-3sin\theta sin2\beta\right] \tag{21}$$

where θ, β are treated as independent variables. Comparing Eqs. (21) and (1), it is obtained that

$$\sigma_{\theta\theta} = \frac{\sigma\sqrt{\pi a}}{\sqrt{2\pi r}} F(\theta, \beta) \tag{22}$$

According to the process of obtaining the fracture angle θ_0, we know

$$\left.\frac{\partial \sigma_{\theta\theta}}{\partial \theta}\right|_{\theta=\theta_0} = 0$$

From the Eq. (22), we have

$$\left.\frac{\partial F(\theta, \beta)}{\partial \theta}\right|_{\theta=\theta_0} = f(\theta_0, \beta) = 0 \tag{23}$$

It is assumed that the equivalent fracture toughness of the I-II mixed crack K_J reaches its minimum value when $\beta = \beta_m$, namely

$$\frac{dK_J}{d\beta} = 0 \quad \text{and} \quad \frac{d^2K_J}{d\beta^2} = 0 \tag{24}$$

Based on the equation (20), it is obtained that

$$\frac{dK_J}{d\beta} = -\frac{K_{\mathrm{IC}}}{F^2(\theta_0, \beta)} \frac{dF(\theta_0, \beta)}{d\beta} = = -\frac{K_{\mathrm{IC}}}{F^2(\theta_0, \beta)} \left(\frac{\partial F(\theta_0, \beta)}{d\theta_0} \frac{\partial \theta_0}{\partial \beta} + \frac{\partial F(\theta_0, \beta)}{d\beta} \right) \tag{25}$$

$$\frac{d^2K_J}{d\beta^2} = -K_{\mathrm{IC}} \left[-\frac{2}{F^3} \left(\frac{dF(\theta_0, \beta)}{d\beta} \right)^2 + \frac{1}{F^2} \frac{d^2F(\theta_0, \beta)}{d\beta^2} \right] = \frac{K_{\mathrm{IC}}}{F^2} \left\{ \frac{2}{F} \left(\frac{dF}{d\beta} \right)^2 - \left[\begin{gathered} \frac{\partial^2 F}{\partial \theta_0{}^2} \left(\frac{\partial \theta_0}{\partial \beta} \right)^2 + 2 \frac{\partial^2 F}{\partial \theta_0 \partial \beta} \frac{\partial \theta_0}{\partial \beta} + \\ + \frac{\partial F}{\partial \theta_0} \frac{\partial^2 \theta_0}{\partial \beta^2} + \frac{\partial^2 F}{\partial \beta^2} \end{gathered} \right] \right\} \tag{26}$$

In Eq. (25), $\partial F(\theta_0, \beta) / \partial \theta_0 = f(\theta_0, \beta) = 0$, and if $dK_J / d\beta = 0$ according to the extremum principle, the Eq. (25) can be written as

$$\frac{dF(\theta_0, \beta)}{d\beta} = 0 \tag{27}$$

Equivalently, it is

$$\left.\frac{\partial F(\theta, \beta)}{\partial \beta}\right|_{\theta=\theta_0} = 0 \tag{28}$$

Combining Eqs. (21) and (28), we obtain

$$\left.\frac{\partial F(\theta, \beta)}{\partial \beta}\right|_{\theta=\theta_0} = \frac{1}{2} cos \frac{\theta_0}{2} \times \times \left[(1 + cos\theta_0)(1 - cos2\beta) - 3sin\theta_0 sin2\beta \right] = 0 \tag{29}$$

As mentioned in above section, the roots of $cos\theta/2 = 0$ can not satisfy the condition of maximum value for $\sigma_{\theta\theta}$. Therefore, only the following expression could be obtained from Eq. (29)

$$(1 + cos\theta_0) sin2\beta - 3sin\theta_0 cos2\beta = 0 \tag{30}$$

Namely,

$$tan2\beta = \frac{3sin\theta_0}{1 + cos\theta_0} \tag{31}$$

Combining Eqs. (16) and (31), it can be obtained that

$$sin\theta_0 \left(12cos^2\theta_0 - 11cos\theta_0 + 1 \right) = 0 \tag{32}$$

The equations to determine the roots of Eq. (32) are

$$sin\theta_0 = 0 \tag{33}$$

$$sin\theta_0 \left(12cos^2\theta_0 - 11cos\theta_0 + 1 \right) = 0 \tag{34}$$

The first root can be obtained from the Eq. (33), it is $\theta_{01} = 0°$. Another two roots can be determined from Eq. (34), they are $\theta_{02} = -35.48°$ and $\theta_{03} = -84.13°$. From Fig. 2 we know that $|\theta_0| < 70°32'$. Therefore, the θ_{03} can't satisfy the condition of maximum value for $\sigma_{\theta\theta}$. It should be rejected. Following, the roots of $\theta_{01} = 0°$ and $\theta_{02} = -35.48°$ will be verified.

Differentiating on the both sides of Eq. (16), it is obtained that

$$sec^2\beta = \frac{3 - cos\theta_0}{sin^2\theta_0} \frac{d\theta_0}{d\beta} \tag{35}$$

Substituting Eq. (16) into Eq. (35)

$$\frac{d\theta_0}{d\beta} = \frac{sin^2\theta_0 \left(1 + tan^2\beta \right)}{3 - cos\theta_0} = \frac{2\left(1 - 3cos\theta_0 + 4cos^2\theta_0 \right)}{3 - cos\theta_0} \tag{36}$$

From Eq. (29), we can determine the terms $\frac{\partial^2 F}{\partial \beta \partial \theta_0}$ and $\frac{\partial^2 F}{\partial \beta^2}$ in Eq. (26)

$$\frac{\partial^2 F}{\partial \beta \partial \theta_0} = \left.\frac{\partial^2 F}{\partial \beta \partial \theta}\right|_{\theta=\theta_0} = -\frac{1}{2} cos \frac{\theta_0}{2} \times \times \left[sin\theta_0 sin2\beta + 3cos\theta_0 cos2\beta \right] \tag{37}$$

$$\frac{\partial^2 F}{\partial \beta^2} = cos \frac{\theta}{2} \left[(1 + cos\theta) cos2\beta + 3sin\theta sin2\beta \right] \tag{38}$$

Similarly, from Eq. (21), the term $\frac{\partial^2 F}{\partial \theta_0^{\ 2}}$ in Eq. (26) can be determined as

$$\frac{\partial^2 F}{\partial \theta_0^{\ 2}} = \left.\frac{\partial^2 F}{\partial \theta^2}\right|_{\theta=\theta_0} = -\frac{3(1-cos2\beta)}{8} cos\frac{\theta_0}{2}\frac{(3-cos\theta_0)}{(3cos\theta_0-1)} \tag{39}$$

Substituting Eqs. (37), (38) and (39) into Eq. (26), we obtain

$$\frac{d^2 K_J}{d\beta^2} = \frac{K_{\mathrm{IC}}}{F^2} cos\frac{\theta_0}{2}\left\{\begin{array}{l}\frac{3}{2}(1-cos2\beta)(3cos\theta_0-1)\frac{(1-3cos\theta_0+4cos^2\theta_0)^2}{(3-cos\theta_0)}+2(sin\theta_0 sin2\beta+3cos\theta_0 cos2\beta)\times \\ \times\frac{(1-3cos\theta_0+4cos^2\theta_0)}{(3-cos\theta_0)}-\left[(1+cos\theta_0)cos2\beta+3sin\theta_0 sin2\beta\right]\end{array}\right\} \tag{40}$$

Following, the two roots of $\theta_0 = 0°$ and $\theta_{02} = 35.48°$ is verified respectively.

(1) θ_0=0°

Substituting $\theta_0 = 0°$ into Eq. (16), the β is determined as 90º. Substituting $\theta_0 = 0°$ and $\beta = \pi/2$ into Eq. (15), we know that $K_{\mathrm{II}} = 0$ (pure mode I crack) under such conditions. And substituting $\theta_0 = 0°$ and $\beta = \pi/2$ into Eq. (40), it is obtained

$$\left.\frac{d^2 K_J}{d\beta^2}\right|_{\substack{\theta=\theta_0=0\\ \beta=\pi/2}} = -2K_{\mathrm{IC}} < 0$$

Obviously, the above expression can not satisfy the condition of $d^2K_J/d\beta^2 > 0$. Therefore, $\beta = \pi/2$, $\theta_0 = 0°$ are not the solution which making the K_J reach its minimum value. Substituting $\beta = \pi/2$, $\theta_0 = 0°$ into Eq. (19), we get

$$K_{J1} = \sigma_c\sqrt{\pi a} = K_{\mathrm{IC}} \quad (\beta = \pi/2,\ \theta_0 = 0) \tag{41}$$

If the design tensile strength of materials $\sigma = \sigma_c$ is given, the maximum critical length of crack in the direction of $\beta = \pi/2$ is

$$a_0 = \frac{K_{\mathrm{IC}}}{\pi\sigma_c^2} \tag{42}$$

(2) θ_{02} = -35.48°

Substituting $\theta_{02} = -35.48°$ into Eq. (16), the β is determined as 68.09º. Substituting $\theta_{02} = -35.48°$ and $\beta = 68.09°$ into Eq. (40), it is obtained

$$\left.\frac{d^2 K_J}{d\beta^2}\right|_{\substack{\theta_0=-35.48\\ \beta=68.09}} = 5.26\frac{K_{\mathrm{IC}}}{F^2}cos\frac{\theta_0}{2} > 0$$

Obviously, the solution of $\theta_{02} = -35.48°$ and $\beta = 68.09°$ satisfy the condition to let the K_J reach its minimum value. Substituting $\theta_{02} = -35.48°$ and $\beta = 68.09°$ into Eq. (19), obtaining

$$K_{J2} = \sigma_c\sqrt{\pi a} = 0.97K_{\mathrm{IC}} \tag{43}$$

Comparing Eqs. (43) and (41), we find that $K_{J2} < K_{J1}$. Therefore, this solution indeed can make K_J reach its minimum value. Here, we define the $\beta_m = 68.09°$ is the preponderant crack angle of material under uniaxial tensile stress.

From Eq. (43), if the length of crack is a, the critical uniaxial tensile strength of materials along the direction of preponderant crack angle is

$$\sigma_c = \frac{0.97K_{\mathrm{IC}}}{\sqrt{\pi a}} \tag{44}$$

Comparing Eqs. (43) and (41), it is found that the uniaxil tensile strength of materials if the direction of crack is $\beta_m = 68.09°$ is smaller about 3% than that of the direction of crack is $\beta = \pi/2$**.**

From Eq. (43), if the design tensile strength of materials $\sigma = \sigma_c$ is given, the critical crack length along the direction $\beta_m = 68.09°$ is

$$a_0 = \frac{0.94K_{\mathrm{IC}}^2}{\pi\sigma_c^2} \tag{45}$$

In engineering, if the direction of force applied is variable or uncertain, and the design tensile strength of materials $\sigma = \sigma_c$ is given, then the results determined by Eq. (45) could be considered as the permitted maximum crack length at arbitrary direction in engineering structural materials. This permitted maximum crack length in structural materials could provide reliable theoretical basis for the detecting and limiting the crack length in structural design. It is noted that this result is obtained based on the brittle fracture instability. The subcritical crack propagation and fatigue fracture and other factors have not been considered.

The Eq. (19) can be rewritten as following form

$$K_J = \frac{K_{\mathrm{IC}}}{\frac{1}{2}sin\beta cos\frac{\theta_0}{2}\left[(1+cos\theta_0)sin\beta-3sin\theta_0 cos\beta\right]} \tag{46}$$

In above equation, the values of β making $K_J \to \infty$ are determined by

$$sin\beta\left[(1+cos\theta_0)sin\beta-3sin\theta_0 cos\beta\right] = 0 \tag{47}$$

One of the solutions of Eq. (47) is $sin\beta = 0$ ($\beta = 0$). The crack is vertical and parallel with the far-field tensile stress. Under such condition, $K_I = K_{II} = 0$; and the crack has no any influence on the stress field near the crack.

Another solution of Eq. (47) is

$$\left(1+cos\theta_0\right)sin\beta-3sin\theta_0 cos\beta=0$$

Namely

$$tan\beta=\frac{3sin\theta_0}{1+cos\theta_0}$$

Obviously, both the β and θ_0 are greater than 0. This result completely can not satisfy the assumption of problem and the experimental data; because if the β is greater than 0 in the problem, then the θ_0 should be smaller than 0 according to the stresses analysis and experiment. Therefore, this solution only exists in mathematics. Actually, there is no any physical meaning.

4. Conclusion and discussion

1. In this study, the brittle tensile capacity and the tensile strength of engineering materials for the I-II mixed crack is derived under uniaxis tensile stress based on the linear elastic maximum circumferential stress theory, see Eqs. (17), (18) and (19).

2. The preponderant fracture angle for the propagation of unstable crack is $\beta_m = 68.09°$ for the I-II mixed crack under uniaxis tensile stress; and the corresponding brittle tensile capacity is $K_J = 0.97K_{IC}$, tensile strength is $\sigma_c = 0.97K_{IC}\big/\sqrt{\pi a}$. From this conclusion, it is proposed that the maximum crack length in engineering materials shouldn't be greater than $a_0 = 0.94K_{IC}^2\big/\pi\sigma_c^2$, if the design tensile strength σ_c is given. This limitation for the crack length in engineering materials provide reliable theoretical basis for detecting the crack length in materials.

3. The theoretical results proposed in this study are different from that of flat elliptical crack model (Jaeger and Cook, 1979), and that of *S* criterion, in which the preponderant crack angle is determined as $\beta_m = 90°$. As stated in this context, the difference of the brittle tensile capacity and the tensile strength for the two types of preponderant crack angle β_m (90° and 68.09°) is not greater than 3%. It is relatively difficult to check the theoretical results using experiment methods. However, the great effort will be performed in the further study to verify the theoretic solution proposed in this study.

4. The evolutionary point of the maximum value of K_J is $\beta_\infty = 0°$. At this moment, the crack is vertical, and parallel with the far-field tensile stress. The crack has no any influence on the stress field in infinite plate.

References

1. **He, X. H.**; **Sun, G. Y.**; **Xue, J. L.** 1997. Study on the fracture behaviour of I-II mixed mode crack of a low carbon alloy, Experiment Research 14(2): 99-103.
2. **Chen, Z. T.**; **Wang, D.** 1994. A mixed fracture criterion for mixed mode crack propagation, Journal of Haerbin Institute of Technology 26(4): 127-131.
3. **Xu, B.**; **Jiang, J.J.** 1995**.** Study of I-II mixed-mode fracture criteria for concrete, Engineering Mechanics 12(2): 13-21.
4. **Erdogan, F.**; **Sih, G.C.** 1963. On the crack extension in plates under plane loading and transverse shear, Journal of Basic Engineering 85: 519-527. http://dx.doi.org/10.1115/1.3656897
5. **Palanaswamy, K.; Knauss, W.G.** 1972. Propagation of a crack under general in-plane tension, International Journal of Fracture Mechanics 8: 114-117. http://dx.doi.org/10.1007/BF00185207
6. **Deng Z.C.** 1999. The maximum circumferential strain fracture criterion for mixed mode of concrete, The Journal of Northwest Agriculture University 27(1): 43-46.
7. **Sih, G.C.** 1974. Stain energy density factor applied to mixed mode crack problems, International Journal of Fracture 10: 305-321. http://dx.doi.org/10.1007/BF00035493
8. **Kohini, K.**; **Marangoni, R.D.**; **Dorogy, G.M.**; **Ezzat, H.A.** 1987. Crack propagation under mode II loading: An effective stress intensity factor method, Engineering Fracture Mechanics 28(1): 93-100. http://dx.doi.org/10.1016/0013-7944(87)90123-8
9. **Theocatis, P.S.**; **Andrianopoulos, N.P.** 1984. Author's closure on the discussion by G. C. Sih and E. E. Gdoutors of the Mises elastic-plastic boundary as the core region in the fracture criteria, Engineering Fracture Mechanics 20: 691-694. http://dx.doi.org/10.1016/0013-7944(84)90044-4
10. **Yehia, N.A.B.** 1991. Distortional strain energy density criterion: the Y-criterion, Engineering Fracture Mechanics 39(3): 477-485. http://dx.doi.org/10.1016/0013-7944(91)90060-E
11. **Hellen, T.K.; Blackburn, W.S.** 1975. The calculation of stress intensity factors for combined tensile and shear loading, International Journal of Fracture 11: 605-617. http://dx.doi.org/10.1007/BF00116368
12. **Liu, X.M.**; **Liu, Y.H.**; **Liang, Y.C.** 2004. Maximum circumferential stress criterion of brittle fracture for V-notch, Journal of Hefei University of Technology 27(9): 1043-1046.
13. **Liu, Y.H.**; **Cui, S.W.**; **Chen, J.G.** 2007. Maximum circumferential stress criterion of brittle fracture at the interface edge of bonded materials, Journal of Hefei University of Technology 30(2): 200-202.
14. **Žiliukas, A.**; **Surantas, A.**; **Žiogas, G.** 2010. Strength and fracture criteria application in stress concentrators areas, Mechanika 3(83): 17-20.
15. **Nazari, M.B.**; **Shariati, M.**; **Eslami, M.R.**; **Hassani B.** 2010**.** Meshless analysis of cracked functionally graded materials under thermal shock, Mechanika 4(84): 20-27.
16. **Shariati, M.**; **Sedighi, M.**; **Saemi, J.**; **Eipakchi, H.R.**; **Allahbakhsh, H.R.** 2010. Numerical and experimental investigation on ultimate strength of cracked cylindrical shells subjected to combined loading, Mechanika 4(84): 12-19.
17. **Wang, D.** 1989. Fracture Mechanics, Haerbin: Haerbin Institute of Technology Press, 104p. (in Chinese).

Zhang Yan, Ji Hongguang, Ye Jianhong, Li Shiyu

TRAPIOJO ATSPARUMO TEMPIANT IR ATITINKAMO LEISTINOJO PLYŠIO KAMPO, ESANT VIENAAŠIAM APKROVIMUI BEI I-II TIPO MIŠRIAM PLYŠIUI, TEORINIS SPRENDIMAS, PAGRĮSTAS MAKSIMALIŲ ŽIEDINIŲ ĮTEMPIŲ KRITERIJUMI

R e z i u m ė

I ir II tipų mišrus plyšys yra vienas iš dažniausiai inžinerinėse medžiagose susidarančių plyšių. Šiame darbe, remiantis maksimalių apskritiminių įtempių kriterijumi yra nustatytas inžinerinių medžiagų, turinčių I ir II tipų mišrių plyšių (plyšio kampas β), trapiojo irimo esant vienaašiam tempimui, intensyvumas K_J, stiprumo riba σ_c ir pradinis irimo kampas θ_0. Papildomai nustatyta, kad ribinėmis sąlygomis tai pačiai konstrukcijai leistinas plyšio kampas, kuriam esant lengviausiai vyksta irimas ir plinta plyšys, yra $\beta = 68.09°$, atitinkamas trapiojo irimo tamprumo intensyvumas $K_J = 0.97K_{IC}$ ir stiprumo riba $\sigma_c = 0.97K_{IC}/\sqrt{\pi a}$. Remiantis pateiktais duomenimis, nustatyta, kad projektuojant inžinerines konstrukcijas, jeigu duota medžiagos stiprumo riba σ_c leistinas maksimalus plyšio ilgis a_0 apribotas $0.94K_{IC}^2/\pi\sigma_c^2$ vadovaujantis teoriniais plyšio ilgio apribojimo pagrindais.

Zhang Yan, Ji Hongguang, Ye Jianhong, Li Shiyu

ANALYTICAL SOLUTIONS OF THE TENSILE STRENGTH AND PREPONDERANT CRACK ANGLE FOR THE I-II MIXED CRACK IN BRITTLE MATERIAL

S u m m a r y

The I-II mixed crack is one of the most common cracks contained in the engineering materials. In this study, the theoretical solutions of the brittle tensile failure capability K_J , the tensile strength σ_c, and the initial fracture angle θ_0 are developed for the engineering materials containing the mode I-II mixed cracks (the crack angle is β) under uniaxial tensile stress loading based on the maximum circumferential stress criterion. Additionally, under the same frame and boundary conditions, it is concluded that the preponderant crack angle β is 68.09 which making the cracks are most easy to fail and propagate; the corresponding brittle tensile capability K_J is $0.97K_{IC}$, and the tensile strength σ_c is $\sigma_c = 0.97K_{IC}/\sqrt{\pi a}$. Based on the results stated above, it is further concluded that, in the engineering structure design, if the tensile strength of materials σ_c is given, the allowed maximum length of cracks a_0 is limited to $0.94K_{IC}^2/\pi\sigma_c^2$, which would provide the theoretical foundations for limiting the maximum length of cracks, and detecting cracks in engineering design.

Keywords: crack, brittle material, tensile strength.

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