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Finite element and isogeometric correlation analysis using modal assurance criterion

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1. Introduction

J.R. Hughes has recently proposed a modelling method allows linking the Computer Aided Design CAD and finite element analysis FEA [1]. The name of the analysis "isogeometric" means that the same basis functions can be used in CAD and FEA.

The main goal of our work is to measure or define the difference between the classical finite element analysis and the new isogeometric analysis in the mechanical behaviour of a thin plate.

The thrust of the isogeometric analysis is to bring the finite element CAD modelling analysis by exploiting the geometric model as a support for the calculation. This can be done through the development of new types of finite element models using the same basis functions as those used in CAD models defining exact geometry. At present, the majority modelling tools uses NURBS functions (Non Uniform Rational B-Spline) for the geometric description. These have interesting properties and stable algorithms to generate and manipulate models. This technique appears promising and subject of numerous papers [2-6]. The use of exact geometry opens interesting perspectives for the problems of locks and propagation of singularities in the numerical computation of thin elastic shells.

In this work, we will detail these functions as well as the description of geometries using NURBS model. Then we explain the principle of finite element analysis and the isogeometric analysis using NURBS basis functions. In the second part we develop a two-dimensional mesh that we will explore with the two methods, the classical finite element and isogemetric finite element.

To conclude, we compare the results obtained using the MAC (Modal Assurance Criterion) correlation function in structural dynamics [7, 8].

2. Isogeometric analysis

Isogeometric finite elements are mainly based on NURBS functions to represent the geometry and the shape as functions. A major advantage of these functions is their continuity which is greater than that used in conventional finite elements method.

2.1. B-Spline basis functions

B-Splines are piecewise polynomial functions with a given class of continuity. They are constructed from a nodes vector which is determined from a set of parametric coordinates $\Xi = \{\xi_1, \xi_2, \ldots, \xi_{n+p+1}\}$, where $\xi_i \in \mathrm{R}$ is the i-th node and i the index, $i = 1, 2, \ldots, n + p + 1$, p is the polynomial order and n the number of B-Spline basis functions. The functions of any order p are defined recursively using the Cox-de Boor formula using the node vector Ξ [9].

We begin by defining the piecewise constant functions ($p = 0$):

$$N_{i,0}(\xi) = \begin{cases} 1 & \text{if } \xi_i \le \xi < \xi_{i+1} \\ 0 & \text{else} \end{cases}, \tag{1}$$

For $p \ge 1$, the recurrence relation is (2):

$$N_{i,p}(\xi) = \frac{\xi - \xi_i}{\xi_{i+p} - \xi_i} N_{i,p-1}(\xi) + \frac{\xi_{i+p+1} - \xi}{\xi_{i+p+1} - \xi_i} N_{i+1,p-1}(\xi), \tag{2}$$

The denominator of many of these functions can become zero in the presence of repeated nodes, when this happens, the convention $\frac{x}{0} = 0$ is adopted.

2.2. Representation of the geometry

The geometrical model $C(\xi)$ can be expressed by a linear combination of the NURBS basis functions R_i and the control points B_i, leading to [9] :

$$C(\xi) = \sum_{i=1}^{n} R_i(\xi) B_i ; \tag{3}$$

$$R_{i,p}(\xi) = \frac{N_{i,p}(\xi) w_i}{W(\xi)} = \frac{N_{i,p}(\xi) w_i}{\sum_{j=1}^{n} N_{j,p}(\xi) w_j}, \tag{4}$$

where ξ is a curvilinear parameter, $R_{i,p}(\xi)$ are NURBS basis functions, and $N_{i,p}(\xi)$ are the B-Spline basis functions.

For a two-dimensional geometry we need two NURBS basis functions like:

$$C(\xi,\eta) = \sum_{i=1}^{n}\sum_{j=1}^{m} R_{i,p}(\xi) R_{j,q}(\eta) B_{i,j}, \tag{5}$$

ξ and η are the curvilinear parameters in x and y directions, $R_{i,p}(\xi)$ and $R_{j,q}(\eta)$ are NURBS basis functions.

Fig. 1 presents a two-dimensional geometry which is a thin plate with its control points.

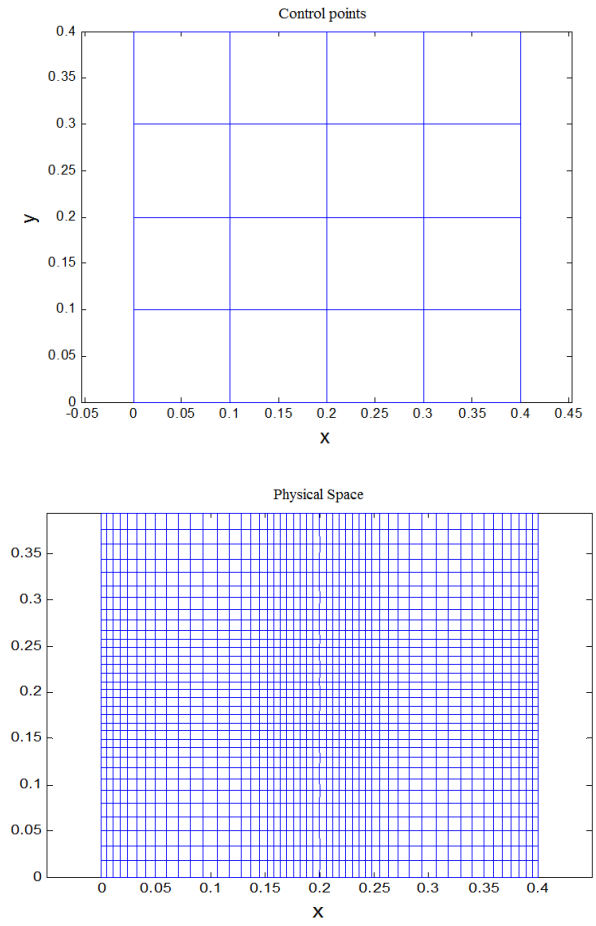

Fig. 1 Control points and NURBS geometry of a thin plate

3. Application and comparison of the models

Generally, the correlation functions are used to compare the eigenmodes (or frequencies) obtained by an analytical method with those obtained experimentally. In our case we will use it to see the correspondence level between the results obtained by the classical finite element method and isogeometric finite element analysis methods.

For this purpose we explore an application of these methods on a two-dimensional model which is a thin plate considered in structural dynamic. At the end of the calculations we will make a comparison of the results by the MAC criterion [7, 8].

Consider the material properties and dimensions of the plate as:

The material of the plate is isotropic.
Young's modulus $E = 21*10^4$ Pa.
Density $\rho = 7890$ kg/m3.
The thickness of the plate $e = 1$ mm.
Length and large of the plate are $a = b = 0.4$ m.

3.1. Finite element model application

3.1.1. Geometrical discretization

The equations used for the discretization forms are those of the rectangular element. In the first case Lagrange Q4 element which represents the first order of the Lagrange polynomials is used and the second case by using the Lagrange Q9 element which represents the second order of Lagrange polynomials.

Determination of the elementary matrices

The mass and stiffness matrices are obtained from the following integral forms [10, 11]:

$$K_e = \int_{-1}^{1}\int_{-1}^{1} eB(\varepsilon,n)^T DB(\varepsilon,n)\left|det[J]\right| d\varepsilon dn\,; \tag{6}$$

$$M_e = \int_{-1}^{1}\int_{-1}^{1} eN(\varepsilon,n)^T \rho N(\varepsilon,n)\left|det[J]\right| d\varepsilon dn\,. \tag{7}$$

For the Q4 element:

$$D = \frac{E}{1-\upsilon^2}\begin{bmatrix} 1 & \upsilon & 0 \\ \upsilon & 1 & 0 \\ 0 & 0 & \dfrac{1-\upsilon}{2} \end{bmatrix}. \tag{8}$$

$$B = \frac{1}{4}\begin{bmatrix} (n-1) & 0 & (1-n) & 0 & (n+1) & 0 & -(n+1) & 0 \\ 0 & (\varepsilon-1) & 0 & -(\varepsilon+1) & 0 & (\varepsilon+1) & 0 & (1-\varepsilon) \\ (\varepsilon-1) & (n-1) & -(\varepsilon+1) & (1-n) & (\varepsilon+1) & (n+1) & (1-\varepsilon) & -(n+1) \end{bmatrix}; \tag{9}$$

$$N = \frac{1}{4}\begin{bmatrix} n11 & 0 & n13 & 0 & n15 & 0 & n17 & 0 \\ 0 & n22 & 0 & n24 & 0 & n26 & 0 & n28 \end{bmatrix}; \tag{10}$$

$$J = \begin{pmatrix} (n-1) & (1-n) & (n+1) & -(n+1) \\ (\varepsilon-1) & -(\varepsilon+1) & (\varepsilon+1) & (1-\varepsilon) \end{pmatrix}\begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ x_4 & y_4 \end{pmatrix}. \tag{11}$$

with $n11 = n22 = (1-n)(1-\varepsilon)$; $n13 = n24 = (1-n)(1-\varepsilon)$; $n15 = n26 = (1-n)(1-\varepsilon)$; $n17 = n28 = (1-n)(1-\varepsilon)$.

The elementary Jacobian matrix is obtained as follows:

Elementary matrices *Me* and *Ke* are of dimension (8,8), eight rows and eight columns. For the case of the Q9 element the procedure is the same except that the dimensions of the matrices. After distribution of the shape equations derived from the Q9 element, the matrices dimension will be:

B = (3, 18);
N = (2, 18);
J = (2, 2);
Me and *Ke* = (18, 18).

Note: Because of the complexity to analytically

integrate the functions we used the numerical Gaussian quadrature to calculate the integral.

3.1.2. Assembling and resolution

After obtaining the mass and stiffness matrices of each element, they are arranged in overall mass and stiffness matrices. After assembly, we proceed to the resolution.

The eigenvalues and eigenvectors of the model are obtained by solving the modal equation:

$$\left(K-\omega^2 M\right)\psi = 0\,. \tag{12}$$

After calculation and assembly of mass and stiffness matrices, the resolution of the matrix equation Eq. (12) gives the eigenvalues shown in Fig. 2 and Fig. 3.

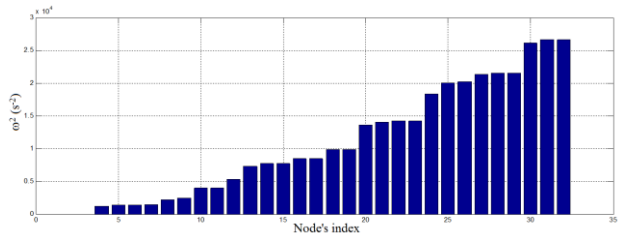

Fig. 2 Eingenvalues of the thin plate using Q4 Lagrange elements

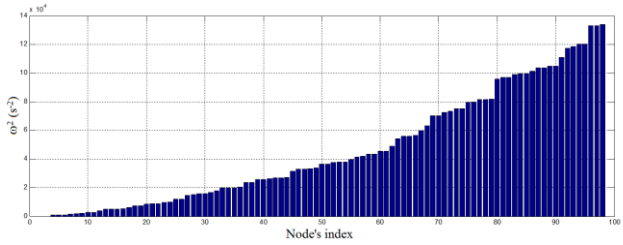

Fig. 3 Eingenvalues of the thin plate using Q9 Lagrange elements

3.2. Isogeometric analysis

The geometry of the model is expressed by a linear combination of NURBS basis functions R_i and control points B_i [9]:

$$C\left(\xi,\eta\right) = \sum_{i=1}^{n}\sum_{j=1}^{m} M_{\,i,p}\left(\xi\right) M_{\,j,q}\left(\eta\right) B_{i,j}\,, \tag{13}$$

$M_{j,q}$ are the NURBS basis functions.

The combination of these basis functions gives

For order 1:

$$D = \frac{E}{1-\upsilon^2}\begin{bmatrix} 1 & \upsilon & 0 \\ \upsilon & 1 & 0 \\ 0 & 0 & \dfrac{1-\upsilon}{2} \end{bmatrix}; \tag{16}$$

the parametric space, subdivided into field called reference element.

For the first order, the basis functions and nodes vectors Ξ = [0 0 0 1/3 2/3 1 1 1] and H = [0 0 0 1/3 2/3 1 1 1] are subdivided into three inter-nodal areas to subdivide the thin plate into nine elements, the same number of elements as in the previous case (conventional finite element) in order to make a comparative study using the correlation function MAC. Fig. 4 illustrates the geometry of the model obtained by the first order NURBS functions and the associated curvilinear parameter space.

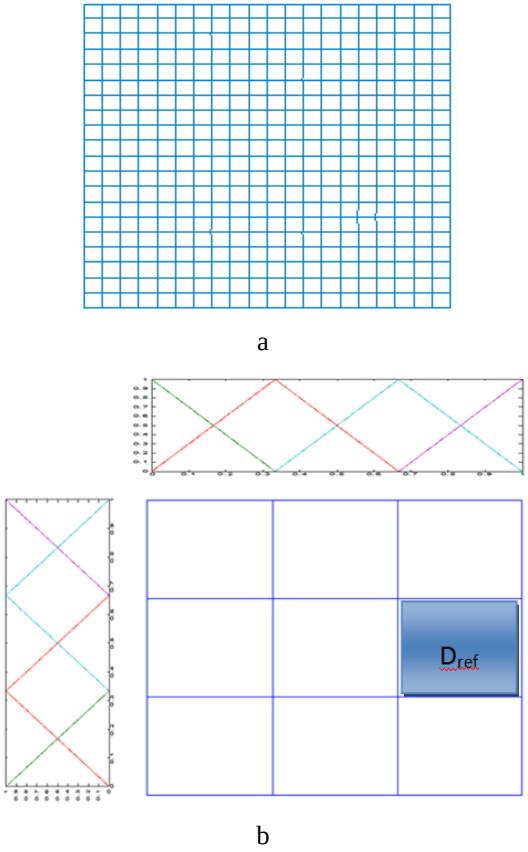

Fig. 4 a - Geometry of the model obtained by first order NURBS functions; b - parameter space associated

Determination of the elementary matrices

The elementary mass and stiffness matrices are obtained from the following integral forms [1]:

$$K_e = \int_0^1\int_0^1 eC\left(\varepsilon,n\right)^T DC\left(\varepsilon,n\right)\left|det\left[J\right]\right| d\varepsilon dn\,; \tag{14}$$

$$M_e = \int_0^1\int_0^1 eM\left(\varepsilon,n\right)^T \rho M\left(\varepsilon,n\right)\left|det\left[J\right]\right| d\varepsilon dn\,. \tag{15}$$

$$C = \frac{1}{4}\begin{bmatrix} (n-1) & 0 & (1-n) & 0 & n & 0 & -n & 0 \\ 0 & (\varepsilon-1) & 0 & -\varepsilon & 0 & \varepsilon & 0 & (1-\varepsilon) \\ (\varepsilon-1) & (n-1) & -\varepsilon & (1-n) & \varepsilon & n & (1-\varepsilon) & -n \end{bmatrix}; \tag{17}$$

$$M = \frac{1}{4}\begin{bmatrix} (1-n)(1-\varepsilon) & 0 & (1-n)\varepsilon & 0 & n\varepsilon & 0 & n(1-\varepsilon) & 0 \\ 0 & (1-n)(1-\varepsilon) & 0 & (1-n)\varepsilon & 0 & n\varepsilon & 0 & n(1-\varepsilon) \end{bmatrix}. \tag{18}$$

The elementary Jacobian matrix is obtained as follows:

$$J = \begin{pmatrix} (n-1) & (1-n) & n & -n \\ (\varepsilon-1) & -\varepsilon & \varepsilon & (1-\varepsilon) \end{pmatrix} \begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ x_4 & y_4 \end{pmatrix}. \tag{19}$$

The *Me* and *Ke* elementary matrices are of dimension (8, 8), eight rows and eight columns, and we used the Gaussian quadrature as in the case of the conventional finite element to calculate the integral.

In the second case, the second order basis functions and the same nodes vectors as the previous are used: Ξ = [0 0 0 1/3 2/3 1 1 1] and H = [0 0 0 1/3 2/3 1 1 1].

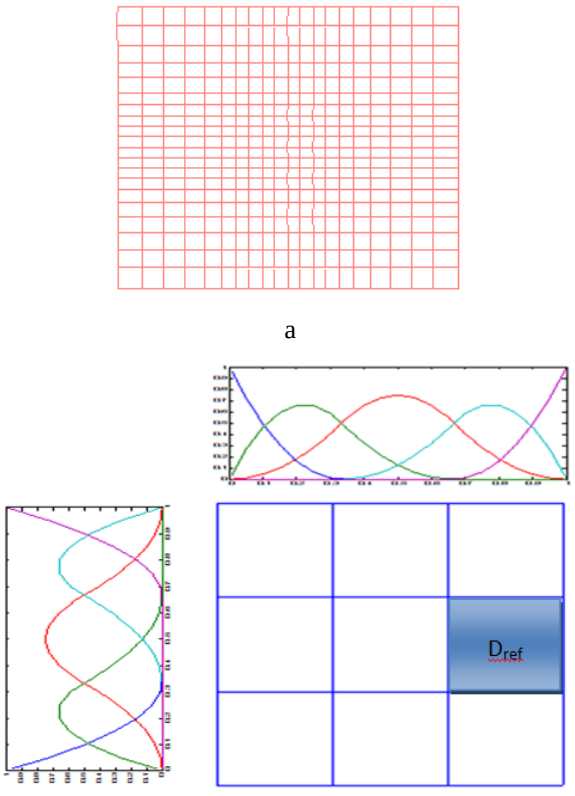

Fig. 5 a - Geometry of the model obtained by second order NURBS functions; b - parameter space associated

The calculation procedure is the same as first order, the difference is in the shape functions and the degrees of freedom number. In order 1, the degrees of freedom number per element is four as Q4 Lagrange element, and the degrees of freedom number in the second order is nine as Q9 Lagrange element.

Fig. 5 illustrates the geometry of the model obtained by second order NURBS functions and the associated curvilinear parameter space.

After calculation and assembly of mass and stiffness matrices, the resolution of the matrix equation Eq. (12) gives the eigenvalues shown in Fig. 6 and Fig. 7.

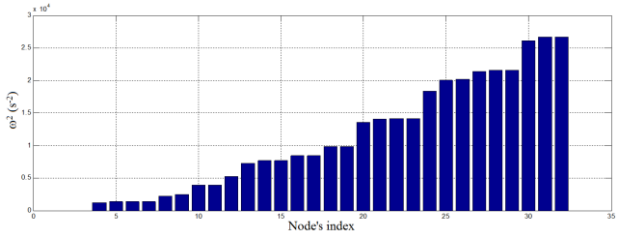

Fig. 6 Eingenvalues of the thin plate using first order NURBS functions

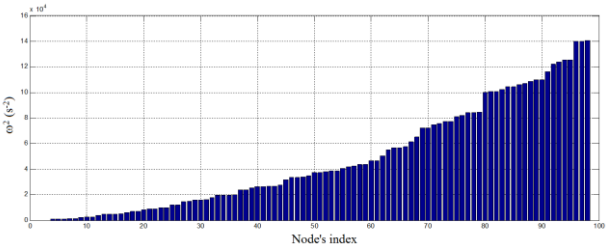

Fig. 7 Eingenvalues of the thin plate using second order NURBS functions

4. Comparison between the two methods

4.1. Modal Assurance criterion

Modal Assurance Criterion (MAC) is a correlation function encountered in matching and comparison of analytical and experimental modes. It is defined as follows [7]:

$$MAC(i,j) = \frac{\left(\{\varphi_i\}^T \{\varphi_j\}\right)^2}{\left(\{\varphi_i\}^T \{\varphi_i\}\right)\left(\{\varphi_j\}^T \{\varphi_j\}\right)}, \tag{20}$$

where $\{\varphi_i\}$ is the i-th eigenmode

The MAC values are in the range [0, 1], a value of 0 indicating complete separation of the two considered

modes, a value of 1 indicates a perfect correlation. A usual criterion is that the representation of a matrix taking the analytical eigenmodes as abscissa and experimental eigenmodes as ordinate. According to this representation, a good correlation between the two compared situations must materialize diagonal high MAC values (> 0.8).

4.2. Comparison using modal assurance criterion

The eigenvalues given by the two modeling methods in the case of first order shape functions are the same Fig. 2 and Fig. 6, this is due to the shape functions that are the same.

For second order Fig. 3 and Fig. 7 the eigenvalues are very close, this slight difference is due to the discretization error of the geometry in the conventional finite element method while the isogeometric analysis gives an exact geometry.

For comparison between the first order Lagrange element Q4 and the first order NURBS function we have computed the MAC matrix which is illustrated by the following figure.

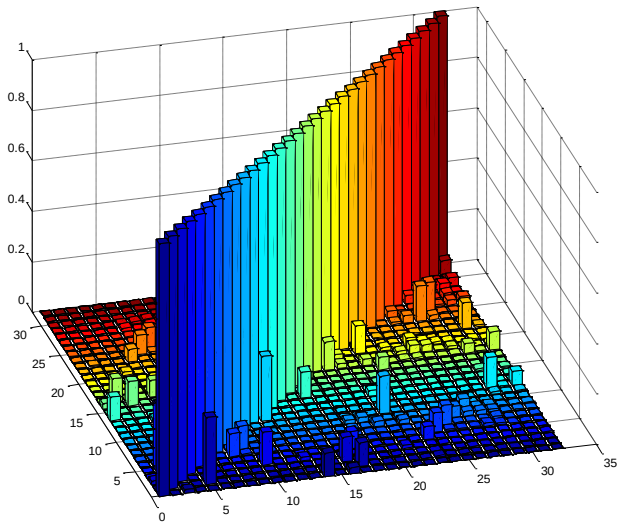

Fig. 8 MAC matrix obtained from the eigenmodes given by first order Lagrange and NURBS functions

All values of the diagonal of the MAC matrix are equal to 1, this means a perfect correlation between the eigenmodes using first order Lagrange polynomials and first order NURBS functions. This means that for first order NURBS basis functions and the Lagrange polynomials are the same, and that's why we got a perfect correlation. The variation of colour from blue to red expresses the variation of the MAC's index from 0 to 32.

For the comparison between second order Lagrange element Q9 and second order NURBS functions we obtain the MAC matrix illustrated in the following figure.

The first values of the diagonal of the MAC matrix are quite large, which means that there is a resemblance between the first modes of the two methods, some of them are equal to 0.98, the difference between the first modes is minimal, but there remains a large number of values lower than 0.8 even they reached values of 0.02, these values indicate a relatively low level similarity between these other modes.

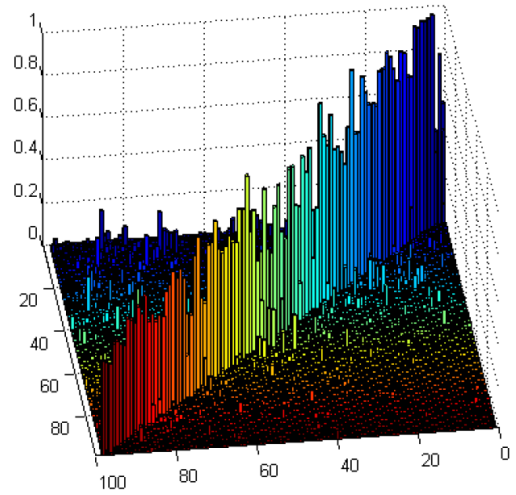

Fig. 9 MAC matrix obtained from the eigenmodes given by second order Lagrange and NURBS functions

5. Coclusions

The theory behind the FEA is quite clear. The elements are distributed directly on a physical domain and we use polynomial basis functions which interpolate the nodal points. It is slightly more difficult to understand the theory behind isogeometric analysis. We must present a parametric space where the elements are defined, and the basis functions are built periodically with convex combinations of B-spline curves, and do not interpolate the control points.

Since the basis functions of isogeometric analysis not interpolate control points, they produce a solution with higher continuity. For first order $p = 1$, the basis functions are equal in both modeling methods. To compare the two methods, we applied these on a two-dimensional model using the MAC criterion we compared the level of matching results (eigenmodes) by both methods, which is the purpose of our work. Because of equal shape functions for $p = 1$, the correlation functions gave a perfect eigenmodes correlation of the two studied methods, second order $p = 2$, the correlation is not good enough, most of the diagonal values of the MAC matrix below 0.8. In conclusion, the difference between the two modeling methods exists, and it seems that the isogeometric analysis technique has high qualities compared to the FEA method, due to the high continuity of the NURBS basis functions and the exact representation of the geometry. Finally, 3D models of structures are well approximated by NURBS and produce a closer representative model of experimental structures than finite element representation.

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FINITE ELEMENT AND ISOGEOMETRIC CORRELATION USING MODAL ASSURANCE CRITERION

S u m m a r y

In the present work we propose to compare the conventional finite element analysis and isogeometric analysis methods. We explore these two modeling methods in the same application in order to identify their differences. From the analytical point of view there is a difference in the type of the shape functions, the Lagrange polynomials used in finite element analysis interpolate the nodal points, and are C0 continuity at the nodal points, in the isogeometric analysis, the NURBS basis functions (Non Uniform Rational B-Spline) have a high continuity and do not interpolate control points. For the comparative study of the two modeling methods, we chose the standard Modal Assurance Criterion (MAC) to compare the eigenmodes. Because of the equality of the first order Lagrange polynomials and the first order NURBS functions, we obtain a perfect eigenmodes correlation of the two methods, but the correlation for the second order shows a slight difference, which highlights a different classification of the two modeling methods.

Keywords: finite element, isogeometric analysis, modal assurance criterion, structural dynamics.

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