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Nonlinear programming and optimal shakedown design of frames

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1. Introduction

Steel frames, which undergo plastic strains and are subjected to variable repeated load, are considered in the paper. Under repeated loading a structure can lose its serviceability because of its progressive plastic failure or because of alternating strain (usually both cases are called cyclic-plastic collapse). The third case when the structure adapts to the existing load and further behaves only elastically is also possible. For civil engineering, the calculation of any complexity elastic–plastic frames subjected to variable repeated load is relevant. Growing number of scientific works dedicated to adapted structure calculation shows importance of these researches [1 - 8]. But there is especially small number of works concerning the optimization of adapted structures under stiffness constraints. This had an influence on the topic of this paper: optimal shakedown design of frames, subjected to variable repeated load, under stiffness constraints. Herein two types of problems can be considered [9]. The first problem is optimal shakedown design of cross-sectional parameters (design problem) and the second one - load optimization problem for a frame subjected to variable repeated load (checking problem). By solving checking problem maximal load variation bounds, ensuring adapted state of the frame and satisfying stiffness requirements of the structure, are to be found.

Solution of frame optimization problems at shakedown is complicated as stress–strain state of dissipative systems depends on loading history [10 - 14]. These difficult optimization problems are implemented applying extremum energy principles and the theory of mathematical programming [15]. That enables to create new iterative algorithm based on Rosen project gradient method [16-19]. Numerical examples of the frames are presented. The results are valid for small displacement assumptions.

2. General mathematical models of optimization problems at shakedown

General mathematical models presented in Table are the basis for the development of optimization mathematical models of frames at shakedown considered in this paper. In both design and checking problems objective functions are described by formulas (1) and (6), where the vectors $\boldsymbol{L}$, $\boldsymbol{T}_{sup}$ and $\boldsymbol{T}_{inf}$ contain coefficients of weight. Yield conditions $\boldsymbol{\varphi}_j$ ($j \in J$) are shown in formulas (2) and (7), where j is the number of all possible combinations $\boldsymbol{F}_j$ of load bounds $\boldsymbol{F}_{sup}$, $\boldsymbol{F}_{inf}$. Formulas (3) and (8) represent complementary slackness conditions of mathematical programming, (4) and (9) are constraints for the problem unknowns. Stiffness constraints are shown in (5) and (10).

Discrete model of the frame at shakedown consists of s ($k = 1, 2, \ldots, s$, $k \in K$) finite elements. Limit force S_{0k} ($k \in K$) is assumed as constant in the whole finite element. The degree of freedom is m, corresponding m - vector of displacements - $\boldsymbol{u}_e = \left(u_{e,1}, u_{e,2}, \ldots, u_{e,m} \right)^T$. Nodal internal forces of the element compound one n – vector of discrete model forces $\boldsymbol{S} = \left(\boldsymbol{S}_1, \boldsymbol{S}_2, \ldots, \boldsymbol{S}_v, \ldots, \boldsymbol{S}_\zeta \right)^T = \left(S_z \right)^T$ and strains – n –vector $\boldsymbol{\Theta} = \left(\boldsymbol{\Theta}_1, \boldsymbol{\Theta}_2, \ldots, \boldsymbol{\Theta}_v, \ldots, \Theta_\zeta \right)^T = \left(\Theta_z \right)^T$,

Table

General mathematical models of optimization problems

Design problem		Checking problem	
find		find	
$\min\ \psi(\boldsymbol{S}_0) = \min\ \boldsymbol{L}^T \boldsymbol{S}_0$	(1)	$\max\ \left(\boldsymbol{T}_{sup}^T \boldsymbol{F}_{sup} + \boldsymbol{T}_{inf}^T \boldsymbol{F}_{inf} \right)$	(6)
subject to		subject to	
$\boldsymbol{\varphi}_j = \boldsymbol{S}_0 - \boldsymbol{\Phi}\left(\boldsymbol{G}\boldsymbol{\lambda} + \boldsymbol{S}_{ej} \right) \geq \boldsymbol{0}$	(2)	$\boldsymbol{\varphi}_j = \boldsymbol{S}_0 - \boldsymbol{\Phi}\left(\boldsymbol{G}\boldsymbol{\lambda} + \boldsymbol{S}_{ej} \right) \geq \boldsymbol{0}$	(7)
$\boldsymbol{\lambda}_j^T \boldsymbol{\varphi}_j = 0,\ \boldsymbol{\lambda}_j \geq \boldsymbol{0}$; $\boldsymbol{\lambda} = \sum_j \boldsymbol{\lambda}_j,\ j \in J$	(3)	$\boldsymbol{\lambda}_j^T \boldsymbol{\varphi}_j = 0,\ \boldsymbol{\lambda}_j \geq \boldsymbol{0}$; $\boldsymbol{\lambda} = \sum_j \boldsymbol{\lambda}_j,\ j \in J$	(8)
$\boldsymbol{S}_0 \geq 0$	(4)	$\boldsymbol{F}_{sup} \geq 0,\quad \boldsymbol{F}_{inf} \geq 0$	(9)
$\boldsymbol{u}_{r,min} \leq \boldsymbol{u}_{r,inf},\quad \boldsymbol{u}_{r,sup} \leq \boldsymbol{u}_{r,max}$	(5)	$\boldsymbol{u}_{r,min} \leq \boldsymbol{u}_{r,inf},\quad \boldsymbol{u}_{r,sup} \leq \boldsymbol{u}_{r,max}$	(10)

$v = 1, 2, \ldots, \zeta$ ($v \in Z$), $z = 1, 2, \ldots, n$. The total number of design sections is ζ.

Load $\boldsymbol{F}(t)$ is characterized by time t, independent variation bounds $\boldsymbol{F}_{sup} = \left(F_{1,sup}, F_{2,sup}, \ \ldots, F_{m,sup}\right)^T$ and $\boldsymbol{F}_{inf} = \left(F_{1,inf}, F_{2,inf}, \ldots, F_{m,inf}\right)^T$ ($\boldsymbol{F}_{inf} \le \boldsymbol{F}(t) \le \boldsymbol{F}_{sup}$). Elastic displacements $\boldsymbol{u}_e(t)$ and forces $\boldsymbol{S}_e(t)$ of the structure are determined using influence matrixes of displacements and forces, $\boldsymbol{\beta} = \left(\boldsymbol{AK}\,\boldsymbol{A}^T\right)^{-1}$, $\boldsymbol{\alpha} = \boldsymbol{KA}^T\boldsymbol{\beta}$, respectively: $\boldsymbol{u}_e(t) = \boldsymbol{\beta}\,\boldsymbol{F}(t)$, $\boldsymbol{S}_e(t) = \boldsymbol{\alpha}\,\boldsymbol{F}(t)$, $\boldsymbol{K} = \boldsymbol{D}^{-1}$. Here $\boldsymbol{A}$ is a coefficient matrix of equilibrium equations $\boldsymbol{A}\,\boldsymbol{S} = \boldsymbol{F}$ and $\boldsymbol{D}$ is a quasi-diagonal flexibility matrix. Residual displacements $\boldsymbol{u}_r$ and forces $\boldsymbol{S}_r$ are related to the vector of plasticity multipliers $\boldsymbol{\lambda}$ by influence matrixes $\boldsymbol{H}$ and $\boldsymbol{G}$: $\boldsymbol{u}_r = \bar{\boldsymbol{H}}\,\boldsymbol{\Phi}^T\boldsymbol{\lambda} = \boldsymbol{H}\,\boldsymbol{\lambda}$, $\boldsymbol{S}_r = \bar{\boldsymbol{G}}\boldsymbol{\Phi}^T\boldsymbol{\lambda} = \boldsymbol{G}\,\boldsymbol{\lambda}$, $\bar{\boldsymbol{H}} = \left(\boldsymbol{AKA}^T\right)^{-1}\boldsymbol{AK}$ and $\bar{\boldsymbol{G}} = \boldsymbol{KA}^T\bar{\boldsymbol{H}} - \boldsymbol{K}$. Here $\boldsymbol{\Phi}$ – the matrix of peace-wise linearized yield conditions $\boldsymbol{\varphi}_j$ (2) and (7). The number of all possible combinations $\boldsymbol{F}_j$ of load bounds $\boldsymbol{F}_{sup}$, $\boldsymbol{F}_{inf}$ is $p = 2^m$ ($\boldsymbol{F}_{inf} \le \boldsymbol{F}_j \le \boldsymbol{F}_{sup}$): $\boldsymbol{S}_{ej} = \boldsymbol{\alpha}\,\boldsymbol{F}_j$, $j = 1, 2, \ldots, p$, ($j \in J$). In the case of two loads F_1, F_2, a domain of elastic force variation (locus) is shown in Fig. 1.

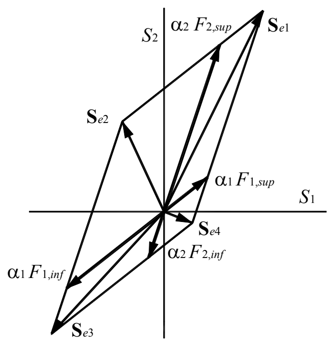


Fig. 1 Locus of elastic forces

Residual displacements $\boldsymbol{u}_r$ of the structure at shakedown can be nonunique: they depend on particular loading history $\boldsymbol{F}(t)$. If load is defined only by variation bounds $\boldsymbol{F}_{inf}$, $\boldsymbol{F}_{sup}$, the calculation of exact values of residual displacements becomes problematical because of unloading phenomenon appearing at cross-sections: then displacements $\boldsymbol{u}_r$ are varying nonmonotonically, it is possible to determine only their lower $\boldsymbol{u}_{r,inf}$ and upper $\boldsymbol{u}_{r,sup}$ variation bounds ($\boldsymbol{u}_{r,inf} \le \boldsymbol{u}_r(t) \le \boldsymbol{u}_{r,sup}$). Stiffness conditions (5) and (10) are realized by the restriction of the structure nodal displacement lower and upper variation bounds $\boldsymbol{u}_{r,min} \le \boldsymbol{u}_{r,inf}$, $\boldsymbol{u}_{r,sup} \le \boldsymbol{u}_{r,max}$.

Mathematical programming theory, the widely used method of the solution of extremum problems, helps not only for the formulation of shakedown problems theory, but also for its solution. Problems (1)-(5) and (6)-(10) can be solved by various computer programs but in this case mechanical interpretation possibilities of optimality criterion of applied algorithms are not revealed. In our works mechanical interpretation of optimality conditions for Rozen algorithm is revealed – it is strain compatibility equations [20].

3. Rozen project gradient method

Rosen project gradient algorithm is universal enough, that it can be applied when objective function and constraints are linear (1) - (5), (6) - (10), or nonlinear [20]. For the optimization problems of volume minimization and determination of maximal load variation bounds containing linear objective function and constraints, application of the Rosen algorithm will be shown. Generally the convex problem of linear programming reads
find

$$max\ \mathscr{F}(\boldsymbol{x}) \tag{11}$$

subject to

$$\varphi_i(\boldsymbol{x}) = \boldsymbol{a}_i{}^T\boldsymbol{x} \le 0\,,\ i = 1,\ 2\,, \ldots ,\ l\,,\ i \in I \tag{12}$$

As function $\varphi_i(\boldsymbol{x})$ is linear, its gradient is $\nabla\varphi_i(\boldsymbol{x}) = \boldsymbol{a}_i$; here $\boldsymbol{a}_i$ is n-vector of multipliers near unknown quantities. In the case of linear constraints (12) gradient matrix of active constraints is noted $\boldsymbol{A}_\kappa$ i.e.

$$\nabla\boldsymbol{\Phi}(\boldsymbol{x}) = \boldsymbol{A}_\kappa = \left[\boldsymbol{a}_1\ \boldsymbol{a}_2\ \ldots\ \boldsymbol{a}_i\ \ldots\ \boldsymbol{a}_\kappa\right] \tag{13}$$

here $\boldsymbol{A}_\kappa$ is $(n \times \kappa)$ – order unit matrix, where n is the measure of Euclidian space E^n and κ is the number of active constraints. Constraints, which are satisfied as equalities, ($\varphi_i\left(\boldsymbol{x}^k\right) = 0$, $i \in I$) are called active ones. Vectors from n-dimensional space, satisfying conditions (12) as equalities, compound $(n \times \kappa)$-order formation noted as G^κ. In Euclidian space E^n movement from $\boldsymbol{x}^k$ is performed in the direction of vector $\boldsymbol{P}_\kappa\nabla\mathscr{F}\left(\boldsymbol{x}^k\right)$ (Fig. 2), which is calculated according to the formula

$$\boldsymbol{P}_\kappa\nabla\mathscr{F}\left(\boldsymbol{x}^k\right) = \left(\boldsymbol{I} - \nabla\boldsymbol{\Phi}\left(\boldsymbol{x}^k\right)\boldsymbol{V}_\kappa\left(\boldsymbol{x}^k\right)\nabla\boldsymbol{\Phi}^T\left(\boldsymbol{x}^k\right)\right)\nabla\mathscr{F}\left(\boldsymbol{x}^k\right) \tag{14}$$

$\boldsymbol{I}$ is $(n \times n)$-order unit matrix, $\nabla\mathscr{F}\left(\boldsymbol{x}^k\right)$ is the gradient of objective function and $(\kappa \times \kappa)$-matrix $\boldsymbol{V}_x\left(\boldsymbol{x}^k\right)$ is expressed as follows: $\boldsymbol{V}_x\left(\boldsymbol{x}^k\right) = \left(\nabla\boldsymbol{\Phi}^T\left(\boldsymbol{x}^k\right)\nabla\boldsymbol{\Phi}\left(\boldsymbol{x}^k\right)\right)^{-1}$. $\boldsymbol{P}_\kappa$ is a projective matrix.

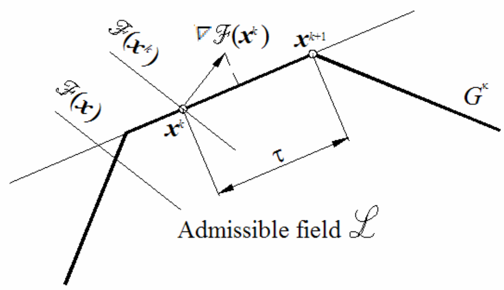


Fig. 2 Rosen algorithm for linear constraints

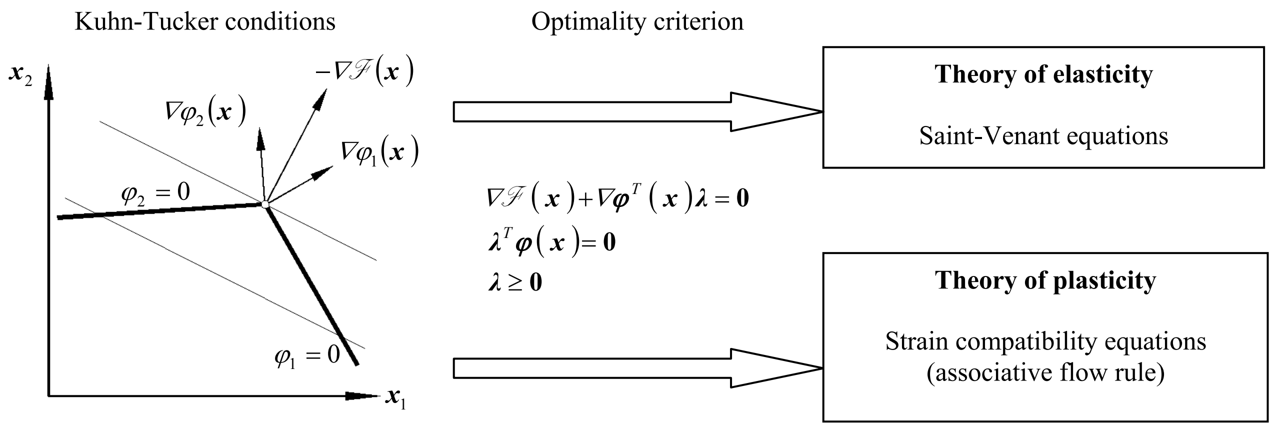


Fig. 3 Kuhn-Tucker optimality conditions are strain compatibility equations of the deformable body mechanic

Vector $\boldsymbol{x}^{k+1} = \boldsymbol{x}^k + \tau' \boldsymbol{P}_\kappa \nabla\mathscr{F}(\boldsymbol{x}^k)$, where $\tau' = min\,\{\tau' | \tau' > 0, \quad i = \kappa+1, \kappa+2,...,l\}$ is the step of the move. Only so vector $\boldsymbol{x}^{k+1}$ "does not leave" admissible field $\mathscr{L} = \{\boldsymbol{x} \,|\, \varphi_i(\boldsymbol{x}) \leq 0, \; i = 1,2,...,l\}$. If the vector does not exist in the admissible range $0 < \tau < \tau'$, for which the magnitude of objective function would be greater than at point $\boldsymbol{x}^{k+1}$ then it is assumed that $\tilde{\boldsymbol{x}}^{k+1} = \boldsymbol{x}^{k+1}$ and the calculation process is continued. If $\nabla\mathscr{F}^T(\boldsymbol{x}^k)\boldsymbol{P}_\kappa\nabla\mathscr{F}(\boldsymbol{x}^{k+1}) < 0$, then the objective function reaches its maximum in the radius between points $\boldsymbol{x}^k$ and $\boldsymbol{x}^{k+1}$. The new size of the step is calculated as follows

$$\tau'' = \tau' \frac{\nabla\mathscr{F}^T(\boldsymbol{x}^k)\boldsymbol{P}_\kappa\nabla\mathscr{F}(\boldsymbol{x}^k)}{\nabla\mathscr{F}^T(\boldsymbol{x}^k)\boldsymbol{P}_\kappa\nabla\mathscr{F}(\boldsymbol{x}^k) - \nabla\mathscr{F}(\boldsymbol{x}^k)\boldsymbol{P}_\kappa\nabla\mathscr{F}(\boldsymbol{x}^{k+1})} \tag{15}$$

In this case $\boldsymbol{x}^{k+1}$ is determined according to the formula: $\boldsymbol{x}^{k+1} = \boldsymbol{x}^k + \tau'' \boldsymbol{P}_\kappa \nabla\mathscr{F}(\boldsymbol{x}^k)$. Vector $\boldsymbol{x}$ is the solution if the following conditions are satisfied

$$\boldsymbol{P}_\kappa \nabla\mathscr{F}(\boldsymbol{x}) = \boldsymbol{0}, \tag{16}$$

$$\boldsymbol{V}_x(\boldsymbol{x})\nabla\boldsymbol{\Phi}^T(\boldsymbol{x})\nabla\mathscr{F}(\boldsymbol{x}) \leq \boldsymbol{0} \tag{17}$$

For correct mechanical interpretation of the conditions (16), it falls to use Kuhn-Tucker conditions [17]. So it is done in the research [20], where it is shown that equation (16) is strain compatibility equation (Fig. 3) and the left side of inequality (17) in absolute value is a vector of plastic multipliers $\boldsymbol{\lambda}$

$$\boldsymbol{\lambda} = \left|\boldsymbol{V}_x(\boldsymbol{x})\nabla\boldsymbol{\Phi}^T(\boldsymbol{x})\nabla\mathscr{F}(\boldsymbol{x})\right| \tag{18}$$

4. Design of minimal volume frame at shakedown

Design of the frame for optimal parameters is performed when yield limit σ_{yk} of the frame material and lengths L_k of its all elements k ($k \in K$) and load variation bounds $\boldsymbol{F}_{sup}$, $\boldsymbol{F}_{inf}$ are known. Depending on the cross-sectional shape various yield conditions can be considered. In this paper, the focus is placed on yield conditions for rolled I steel sections (Fig. 4). Relation $c_k = \dfrac{M_{0k}}{N_{0k}}$, $k \in K$ should be prescribed in advance. Limit moment $M_{0k} = \sigma_{yk} W_{pl,k} = \xi(\sigma_{yk}, A_k)$ and limit axial force $N_{0k} = \sigma_{yk} A_k$ of the element are functions of cross-sectional area A_k and yield limit of material σ_{yk}. True, usually one or the other specific dimension of the cross-section (for instance, flange thickness t_f of I-section while the width of flange b is fixed; see Example 1) participate in functional relation $M_{0k} = \xi(\sigma_{yk}, A_k)$ instead of cross-sectional area A_k. The problem of frame optimal parameters distribution design reads: minimize $\sum_k L_k M_{0k}$, subject to the structure strength and stiffness constraints find

$$min \;\sum_k L_k M_{0k} \tag{19}$$

subject to

$$\boldsymbol{\varphi}_j = \boldsymbol{M}_0 - \boldsymbol{\Phi}(\boldsymbol{G}\boldsymbol{\lambda} + \boldsymbol{S}_{ej}) \geq \boldsymbol{0} \tag{20}$$

$$\sum_j \boldsymbol{\lambda}_j^T \boldsymbol{\varphi}_j = 0, \; \boldsymbol{\lambda}_j \geq \boldsymbol{0}, \; \boldsymbol{\lambda} = \sum_j \boldsymbol{\lambda}_j \tag{21}$$

$$M_{0k,max} \geq M_{0k} \geq M_{0k,min}, \; k \in K, \; j \in J \tag{22}$$

$$\boldsymbol{u}_{min} \leq \boldsymbol{u}_{r,inf}, \quad \boldsymbol{u}_{r,sup} \leq \boldsymbol{u}_{max} \tag{23}$$

Limit moments M_{0k} of the frame elements and vectors of plasticity multipliers $\boldsymbol{\lambda}_j \geq \boldsymbol{0}$, $j \in J$ are unknowns of nonlinear mathematical programming problem (19)-(23). Formulas (21) represent complementary slackness conditions of mathematical programming [21]. Constructive requirements of frames $M_{0k,\,max}$ and $M_{0k,\,min}$ are shown in conditions (22). Problem (19)-(23) is not exactly the volume minimization problem, because limit moments M_{0k} are used in objective function. When volume of the frame is directly included into objective function mathematical model of the frame volume minimization is as follows find

$$min \sum_k L_k A_k \tag{24}$$

subject to

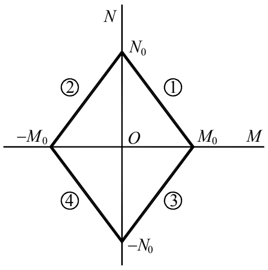
$$\boldsymbol{\varphi}_j = \boldsymbol{M}_0 - \boldsymbol{\Phi}\left(\boldsymbol{G}\,\boldsymbol{\lambda} + \boldsymbol{S}_{ej}\right) \geq \boldsymbol{0} \tag{25}$$

$$\sum_j \boldsymbol{\lambda}_j^T \boldsymbol{\varphi}_j = 0\,,\ \boldsymbol{\lambda}_j \geq \boldsymbol{0}\,,\ \boldsymbol{\lambda} = \sum_j \boldsymbol{\lambda}_j\,,\ j \in J \tag{26}$$

$$A_k \geq A_{k,min}\,,\quad k \in K \tag{27}$$

$$\boldsymbol{u}_{r,min} \leq \boldsymbol{u}_{r,inf}\,,\qquad \boldsymbol{u}_{r,sup} \leq \boldsymbol{u}_{r,max} \tag{28}$$

Cross-sectional areas A_k, $k \in K$ (or other specific dimension of the cross-section) of the frame elements and vectors of plasticity multipliers $\boldsymbol{\lambda}_j \geq \boldsymbol{0}$, $j \in J$ are unknowns of nonlinear mathematical programming problem (24)-(28).



$$\boldsymbol{\Phi}_v = \begin{bmatrix} 1 & c_k \\ -1 & c_k \\ 1 & -c_k \\ -1 & -c_k \end{bmatrix} \qquad v \in Z$$

Fig. 4 Linear yield conditions

Lower bounds of cross-sectional areas $A_{k,min}$ are included into constructive constraints (27) $A_k \geq A_{k,min}$. It is not difficult to introduce elastic displacements into stiffness constraints (28). Limit moments $\boldsymbol{M}_0$ and influence matrixes $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\boldsymbol{G}$, $\boldsymbol{H}$ are related with unknowns A_k, $k \in K$; the listed matrixes are recalculated during solution of the problem (24)-(28). If stiffness constrains are neglected, cyclic-plastic collapse of the frame is reached.

When only bending moments M are taken in to account in the frame calculation, the following mathematical model of the frame volume minimization is obtained
find

$$min \sum_k L_k A_k \tag{29}$$

subject to

$$\boldsymbol{\varphi}_{max} = \boldsymbol{M}_0 - \boldsymbol{G}\,\boldsymbol{\lambda} - \boldsymbol{M}_{e,max} \geq \boldsymbol{0}$$

$$\boldsymbol{\varphi}_{min} = \boldsymbol{M}_0 + \boldsymbol{G}\,\boldsymbol{\lambda} + \boldsymbol{M}_{e,min} \geq \boldsymbol{0} \tag{30}$$

$$\boldsymbol{\lambda}_{max}^T \boldsymbol{\varphi}_{max} = 0\,,\ \boldsymbol{\lambda}_{min}^T \boldsymbol{\varphi}_{min} = 0\,,\ \boldsymbol{\lambda}_{max} \geq \boldsymbol{0}\,,\ \boldsymbol{\lambda}_{min} \geq \boldsymbol{0} \tag{31}$$

$$\boldsymbol{\lambda} = \left(\boldsymbol{\lambda}_{\max}, \boldsymbol{\lambda}_{min}\right)^T \tag{32}$$

$$A_k \geq A_{k,min}\,,\quad k \in K \tag{33}$$

$$\boldsymbol{u}_{r,min} \leq \boldsymbol{u}_{r,inf}\,,\quad \boldsymbol{u}_{r,sup} \leq \boldsymbol{u}_{r,max} \tag{34}$$

Extreme elastic bending moments $\boldsymbol{M}_{e,\max} = \boldsymbol{\alpha}_{sup}\,\boldsymbol{F}_{sup} - \boldsymbol{\alpha}_{inf}\,\boldsymbol{F}_{inf}$, $\boldsymbol{M}_{e,\min} = \boldsymbol{\alpha}_{sup}\,\boldsymbol{F}_{inf} + \boldsymbol{\alpha}_{inf}\boldsymbol{F}_{sup}$ are known in the problem (29)-(34). Matrix $\boldsymbol{\alpha}_{sup}$ is formated in the following way: only positives values are retrieved from the influence matrix $\boldsymbol{\alpha}$, the rest components are set to zero and respectively matrix $\boldsymbol{\alpha}_{inf}$ - only negatives values are retrieved from $\boldsymbol{\alpha}$, the rest components are set to zero. Unknowns are cross-sectional areas A_k, $k \in K$ of the elements and vectors of plasticity multipliers $\boldsymbol{\lambda}_{max}$, $\boldsymbol{\lambda}_{min}$.

In case of monotonically increasing load $j = 1$ and conditions (25), (26) of all discretized frame obtain the following form: $\boldsymbol{\varphi} = \boldsymbol{M}_0 - \boldsymbol{\Phi}\left(\boldsymbol{G}\,\boldsymbol{\lambda} + \boldsymbol{S}_e\right) \geq \boldsymbol{0}$, $\boldsymbol{\lambda}^T \boldsymbol{\varphi} = 0$, $\boldsymbol{\lambda} \geq \boldsymbol{0}$. Stiffness constrains (28) of the frame become more simplified: $\boldsymbol{u}_{r,min} \leq \boldsymbol{H}\,\boldsymbol{\lambda} \leq \boldsymbol{u}_{r,max}$. Scope of the problem (25)-(28) becomes reduced and computer realization of the problem is simpler.

It should be noted that numerical solution of the problems (24)-(28), (29)-(34) is easier when complementary slackness conditions are moved to objective function. Then the problem (29)-(34) obtains the following form [16]
find

$$min\left(\sum_k L_k A_k + \boldsymbol{\lambda}_{max}^T \boldsymbol{\varphi}_{max} + \boldsymbol{\lambda}_{min}^T\,\boldsymbol{\varphi}_{min}\right) \tag{35}$$

subject to

$$\boldsymbol{\varphi}_{max} = \boldsymbol{M}_0 - \boldsymbol{G}\,\boldsymbol{\lambda} - \boldsymbol{M}_{e,max} \geq \boldsymbol{0}$$

$$\boldsymbol{\varphi}_{min} = \boldsymbol{M}_0 + \boldsymbol{G}\,\boldsymbol{\lambda} + \boldsymbol{M}_{e,min} \geq \boldsymbol{0} \tag{36}$$

$$\boldsymbol{\lambda}_{max} \geq \boldsymbol{0}\,,\ \boldsymbol{\lambda}_{min} \geq \boldsymbol{0} \tag{37}$$

$$\boldsymbol{\lambda} = \left(\boldsymbol{\lambda}_{max}, \boldsymbol{\lambda}_{min}\right)^T \tag{38}$$

$$A_k \geq A_{k,min}\,,\quad k \in K \tag{39}$$

$$\boldsymbol{u}_{r,min} \leq \boldsymbol{u}_{r,inf}\,,\quad \boldsymbol{u}_{r,sup} \leq \boldsymbol{u}_{r,max} \tag{40}$$

5. Shakedown load optimization of frames

In the case of variable repeated load, the problem of load variation bound $\boldsymbol{F}_{sup}$, $\boldsymbol{F}_{inf}$ determination is important also. It stated as follows: find shakedown load variation bounds $\boldsymbol{F}_{sup}$, $\boldsymbol{F}_{inf}$, satisfying the prescribed optimality criterion max $\left\{\boldsymbol{T}_{sup}^T\,\boldsymbol{F}_{sup} + \boldsymbol{T}_{inf}^T\,\boldsymbol{F}_{inf}\right\}$, also strength and stiffness requirements of the structure. Here $\boldsymbol{T}_{sup}$, $\boldsymbol{T}_{inf}$ are the optimality criterion weight coefficient vectors.

Then mathematical model of shakedown load optimization problem for the frames reads
find

$$max\left\{\boldsymbol{T}_{sup}^T\,\boldsymbol{F}_{sup} + \boldsymbol{T}_{inf}^T\,\boldsymbol{F}_{inf} - \sum_j \boldsymbol{\lambda}_j^T \boldsymbol{\varphi}_j\right\} \tag{41}$$

subject to

$$\boldsymbol{\varphi}_j = \boldsymbol{M}_0 - \boldsymbol{\Phi}\left(\boldsymbol{G}\,\boldsymbol{\lambda} + \boldsymbol{S}_{ej}\right) \geq \boldsymbol{0} \tag{42}$$

$$\boldsymbol{\lambda}_j \geq \boldsymbol{0}\,,\ \boldsymbol{\lambda} = \sum_j \boldsymbol{\lambda}_j\,,\ j \in J \tag{43}$$

$$\boldsymbol{F}_{sup} \geq \boldsymbol{0}\,,\ \boldsymbol{F}_{inf} \geq \boldsymbol{0} \tag{44}$$

$$\boldsymbol{u}_{r,min} \leq \boldsymbol{u}_{r,inf}\,,\ \boldsymbol{u}_{r,sup} \leq \boldsymbol{u}_{r,max} \tag{45}$$

The vector of limit bending moments $\boldsymbol{M}_0$ and the limits of residual displacements $\boldsymbol{u}_{r,min}$, $\boldsymbol{u}_{r,max}$ are known in the problem (41)-(45). Optimal solution of the problem (41)–(45) is vectors $\boldsymbol{F}^*_{sup}$, $\boldsymbol{F}^*_{inf}$ and $\boldsymbol{\lambda}^*_j$, $j \in J$.

When only bending moments M are taken in to account, the following mathematical model of frame shakedown load optimization is obtained
find

$$max\left\{ \boldsymbol{T}^T_{sup}\, \boldsymbol{F}_{sup} + \boldsymbol{T}^T_{inf}\, \boldsymbol{F}_{inf} - \boldsymbol{\lambda}^T_{max}\, \boldsymbol{\varphi}_{max} - \boldsymbol{\lambda}^T_{min}\, \boldsymbol{\varphi}_{min} \right\} \quad (46)$$

subject to

$$\boldsymbol{\varphi}_{max} = \boldsymbol{M}_0 - \boldsymbol{G}\,\boldsymbol{\lambda} - \boldsymbol{M}_{e,max} \geq \boldsymbol{0}$$

$$\boldsymbol{\varphi}_{min} = \boldsymbol{M}_0 + \boldsymbol{G}\,\boldsymbol{\lambda} + \boldsymbol{M}_{e,min} \geq \boldsymbol{0} \quad (47)$$

$$\boldsymbol{M}_{e,max} = \boldsymbol{\alpha}_{sup}\boldsymbol{F}_{sup} - \boldsymbol{\alpha}_{inf}\boldsymbol{F}_{inf}$$

$$\boldsymbol{M}_{e,min} = -\boldsymbol{\alpha}_{sup}\boldsymbol{F}_{inf} + \boldsymbol{\alpha}_{inf}\boldsymbol{F}_{sup} \quad (48)$$

$$\boldsymbol{F}_{sup} \geq \boldsymbol{0}\,,\; \boldsymbol{F}_{inf} \geq \boldsymbol{0} \quad (49)$$

$$\boldsymbol{\lambda} = \left(\boldsymbol{\lambda}_{max}, \boldsymbol{\lambda}_{min} \right)^T \quad (50)$$

$$\boldsymbol{\lambda}_{max} \geq \boldsymbol{0}\,,\; \boldsymbol{\lambda}_{min} \geq \boldsymbol{0} \quad (51)$$

$$\boldsymbol{u}_{r,min} \leq \boldsymbol{u}_{r,inf}\,,\; \boldsymbol{u}_{r,sup} \leq \boldsymbol{u}_{r,max} \quad (52)$$

Load variation bound $\boldsymbol{F}_{sup}$, $\boldsymbol{F}_{inf}$ and vectors of plasticity multipliers $\boldsymbol{\lambda}_j \geq \boldsymbol{0}$, $j \in J$ are unknowns of nonlinear mathematical programming problem (46)-(52).

6. Numerical examples

6.1. Example 1

The two-storey frame shown in Fig. 5 is subjected by two independent loads: vertical forces of the magnitude $2V$, $3V$ acting in the middle of each beam and horizontal forces $2H$, H. Variation limits of the load are defined by inequalities $0 \leq H \leq H_{sup} = 40\,\text{kN}$, $0 \leq V \leq V_{sup} = 65\,\text{kN}$. The main task is to determine minimal volume of adapted frame (Fig. 5) according to the mathematical models (24)-(28) and (29)-(34), when the frame is made from steel, which elasticity modulus $E = 210$ GPa and the yield limit $\sigma_y = 200$ MPa. Cross-sections of the frame columns and beams are shown in Fig. 6. Parameters b and h' remain the same during all optimization process, only thickness of the flanges is varying. Initial thickness of the flanges is assumed $t^0_{f,col} = 14\,\text{mm}$ for the frame columns and $t^0_{f,beam} = 20\,\text{mm}$ for the beams. Thus, initial cross-sectional areas of the columns and beams are $A^0_{col} = A^0_1 = A^0_2 = 56\,\text{cm}^2$ and $A^0_{beam} = A^0_3 = A^0_4 = 80\,\text{cm}^2$, respectively. Initial volume of the structure is $V^0 = 259200$ cm^3 . Limit forces of cross-sections are calculated according to the following formulas:

$$M_0 = \sigma_y b t h' = \sigma_y A \frac{h'}{2}\,,\; N_0 = \sigma_y 2bt = \sigma_y A$$

Initial limit forces of the columns are $M^0_{0,col} = 160$ kNm and $N^0_{0,col} = 1120$ kN , limit forces of the beams are $M^0_{0,beam} = 320$ kNm and $N^0_{0,beam} = 1600$ kN ; relations $c_{col} = 0.2$ and $c_{beam} = 0.125$. Yeld conditions are aproximated by four lines (coefficients of lines described in matrix $\boldsymbol{\Phi}_v$ are shown in Fig. 4).

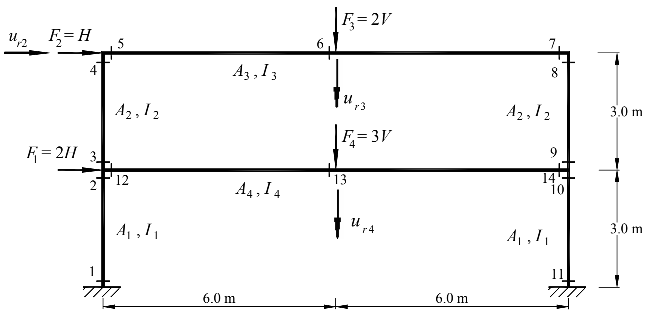


Fig. 5 Discretized frame

Minimal volume searching is performed in the two following cases:

A1 – when the vector of inner forces of discretized frame is $\boldsymbol{S} = \left(\boldsymbol{M}, \boldsymbol{N}\right)^T = \left(M_1, M_2, M_3, \ldots, M_{14}, N_1, N_2, \ldots, N_6\right)^T = \left(S_z\right)^T$, $z = 1, 2, \ldots, n = 20$, i.e. both bending moments M and axial forces N are taken into account.

A2 – when the vector of inner forces of discretized frame is $\boldsymbol{M} = \left(M_z\right)^T = \left(M_1, M_2, M_3, \ldots, M_{14}\right)^T$, $z = 1, 2, \ldots, n = 14$, i.e. only bending moments M are evaluated.

In the case A1 frame volume minimization is performed according to the mathematical model (24)-(28). Unknowns are cross-sectional areas of the frame columns and beams A_k , $k \in K$ and vectors of plasticity multipliers $\boldsymbol{\lambda}_j$, $j = 1, 2, 3$. In the case A2 the frame volume minimization problem is solved using the mathematical model (29)-(34). Unknowns are cross-sectional areas A_k, $k \in K$ and vectors of plasticity multipliers $\boldsymbol{\lambda}_{max}$, $\boldsymbol{\lambda}_{min}$.

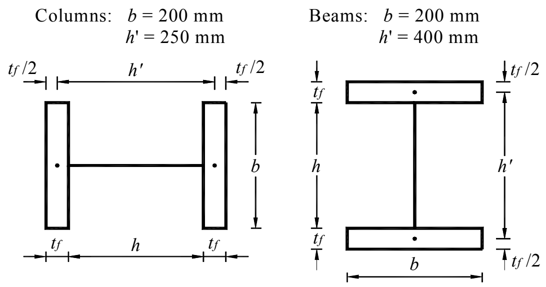


Fig. 6 Geometry of cross-sections

Without any residual displacement constraints (28) or (34), the following minimum volumes of the frame were obtained: $V_{min} = 265288\,\text{cm}^3$ in the case A1 and $V_{min} = 246812\,\text{cm}^3$ in the case A2 (in both cases elastic-plastic state is just before cyclic plastic failure).

Later, the following residual displacement constraints were imposed for displacement $u_{r,2}$ (Fig. 5): $0 \le u_{r,2} \le u_{r,max}$ (here $u_{r,max}$ = 5, 10, 15, 20, 23 mm). Variation of the frame volume depending on prescribed limit on residual displacement $u_{r,max}$ is shown in Fig. 7 for both cases A1 and A2.

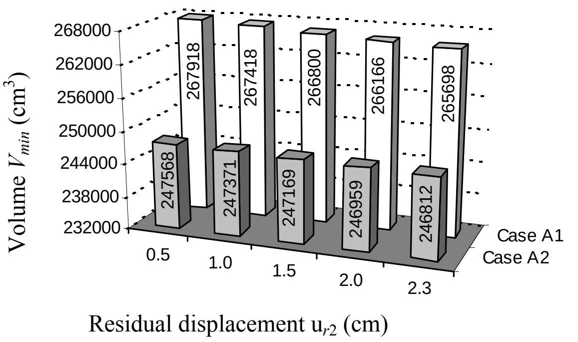


Fig. 7 Variation of minimal volume V_{min} in terms of $u_{r,2}$

6.2 Example 2

The frame is subjected by repeated variable load $0 \le F_2 \le F_{2,sup}$, $0 \le F_3 \le F_{3,sup}$ ir $0 \le F_4 \le F_{4,sup}$. Discretized frame, direction of forces F_2, F_3, F_4 and its application place is shown in Fig. 5. The frame columns HE 300A and beams IPE 450 are made from steel, which elasticity modulus $E = 210\,\text{GPa}$ and yield limit $\sigma_y = 235\,\text{MPa}$. The main task is to determine maximal load variation bounds $F_{2,sup}$, $F_{3,sup}$ and $F_{4,sup}$, i. e. find max $\left(F_{2,sup} + F_{3,sup} + F_{4,sup}\right)$.

Vector of the inner forces of discretized frame (Fig. 5), when bending moments M and axial forces N are taken into account is: $\boldsymbol{S} = \left(\boldsymbol{M}, \boldsymbol{N}\right)^T$ $= \left(M_1, M_2, M_3, ..., M_{14},\ N_1, N_2, ..., N_6\right)^T$ $= \left(S_z\right)^T$ $z = 1, 2, ..., n = 20$. Limit bending moment M_0 and limit axial force N_0 of the columns and beams are calculated according to the following formulas: $M_0 = \sigma_y W_p$, $N_0 = \sigma_y A$.

Load optimization problem max $\left(F_{2,sup} + F_{3,sup} + F_{4,sup}\right)$ is solved according to the mathematical model (41)–(45), when matrix $\boldsymbol{\Phi}_v$, shown in Fig. 4, is taken into account.

Without residual displacement constraints (45) - i.e. in the state near cyclic plastic failure - the following load variation bounds were obtained: $F^*_{2sup} = 257.47\,\text{kN}$, $F^*_{3,sup} = 151.56\,\text{kN}$ and $F^*_{4,sup} = 164.65\,\text{kN}$ (max $\left(F_{2,sup} + F_{3,sup} + F_{4,sup}\right) = 573.68$).

When residual displacement constraints (45) $0 \le u_{r,2} \le u_{r2,max} = 10.0\,\text{mm}$, $0 \le u_{r,3} \le u_{r3,max} = 15.0\,\text{mm}$ and $0 \le u_{r,4} \le u_{r4,max} = 15.0\,\text{mm}$ are evaluated, load variation bounds were obtained: $F_{2sup} = 131.55\,\text{kN}$, $F_{3,sup} = 189.81\,\text{kN}$, $F_{4,sup} = 216.49\,\text{kN}$ (max $\left(F_{2,sup} + F_{3,sup} + F_{4,sup}\right) = 537.85$).

7. Conclusions

1. The main difficulty in solving the problem of determinating the optimal parameter distribution of adapted frame is the reasoning of more realistic relation between the area and limit bending moment of different shape cross-sections. For that purpose it is useful to obtain a correlation between the mentioned quantities.

2. There are created mathematical models of the optimization problem for shakedown frames, which evaluate steel plastic deformations and serviceability requirements.

3. There is created a new algorithm that solves problems, which considers for the displacements non-monotonic variation of shakedown frames.

4. There is presented the possibility to use section databases in the real minimal volume frame design problems.

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NETIESINIS PROGRAMAVIMAS IR RĖMŲ OPTIMIZACIJA PRISITAIKOMUMO SĄLYGOMIS

R e z i u m ė

Straipsnyje nagrinėjama matematinio programavimo teorija, kuri yra plačiai paplitusi kaip ekstreminių uždavinių sprendimo metodas. Ji talkina prisitaikomumo teorijos optimizavimo uždavinių nagrinėjimui nuo jų matematinių modelių sudarymo iki skaitinio sprendinio rezultatų. Bendrieji optimizavimo uždavinių matematiniai modeliai pritaikyti optimalių idealiai tampriai - plastiškai deformuotų rėmų parametrų arba apkrovos pasiskirstymams prisitaikymo būvyje rasti. Uždaviniai spręsti taikant Rozeno projektuojamųjų gradientų metodą. Pateikta šio metodo optimalumo kriterijaus mechaninė interpretacija. Skaitiniai rėmų optimizacijos rezultatai gauti prisilaikant mažų poslinkių prielaidos.

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NONLINEAR PROGRAMMING AND OPTIMAL SHAKEDOWN DESIGN OF FRAMES

S u m m a r y

This paper considers matematical programming theory, which is widely used as a method of extremum problems solution. It helps for the investigation of shakedown problems from creating of it's mathematical models till receiving numerical solution results. Common mathematical models of optimization are adapted to find optimal parameters or load distribution of elastic perfectly-plastic shakedown frames. Rosen project gradient method is applied to solve the problems. Mechanical interpretation of optimality criterion is presented for the mentioned method. Numerical results of frame optimization problems are received with small displacements assumption.

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НЕЛИНЕЙНОЕ ПРОГРАММИРОВАНИЕ И ОПТИМИЗАЦИЯ РАМ В УСЛОВИЯХ ПРИСЛОСОБЛЯЕМОСТИ

Р е з ю м е

Теория математического программирования, широко распространившаяся как метод решения экстремальных задач, сопутствует исследованию задачи теории пластичности от ее постановки до окончательного решения. В статье общие математические модели оптимизации отнесены к определению оптимального распределения параметров или нагруэки идеально упруго-пластических рам в условиях прислособляемости. Для решения полученых нелинейных задач применен метод проектируемых градиентов Розена. Приведена механическая интерпретация критерев оптимальности этого метода. Численые результаты оптимизации рам получены в рамках теории малых перемещений.

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