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Plate buckling under complex loading

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1. Introduction

The investigation of a plate carrying ability covers not only the strength but the buckling also. It is important for the thin and long plates especially. Thin plates of various shapes are often subjected to normal compressive and shearing loads acting in the middle plane of the plate. Under certain condition such loads can result in a plate buckling. The buckling load depends on the plate thickness; the thinner the plate, the lower is the buckling load. In many cases a bending of thin plate elements may be attributed to an elastic instability and not to the lack of their strength. They experience not only the bending in one plane but the lateral bending including the torsion also. Usually these problems are analysed in cases of large dimensions straps [1-5]. But the used load in these cases is uniform. The problems as the bending moment is used alongside with torsion are not analysed.

2. Plate under bending moment

This problem was analysed by Timoshenko [1]. The loading scheme is presented in Fig. 1.

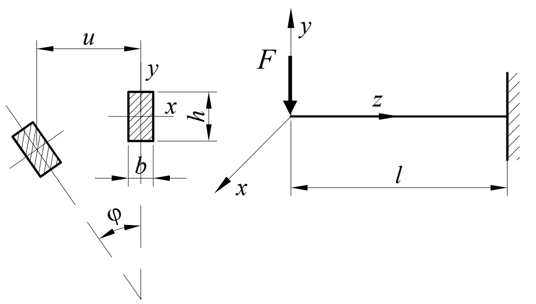

Fig. 1 Cantilever plate loaded by force on free end

The differential equations of bended plate can be written as follows [1]

$$EI_y \frac{d^2u}{dz^2} = -M_x \varphi \tag{1}$$

$$GI_{tor} \frac{d\varphi}{dz} = M_x \frac{du}{dz} + M_z \tag{2}$$

where E is elasticity modulus; G is shear modulus; I_y is axial moment of inertia; I_{tor} is torsion moment of inertia; $M_x = M = F\,z$ (F is force); M_z is torsion moment; x, y, z are coordinates; u is displacement; φ is torsion angle; h, b are dimensions of plate cross-section; l is length of plate.

The torsion moment M_z in Timoshenko solution is equal to zero.

After embedding of $M_x = M$; $M_z = 0$; $EI_y = Ehb^3/12$; $GI_{tor} = Ghb^3/3$; $EI_y = B$; $GI_{tor} = C$; $F^2/BC = k^2$ in Eq. (1) and Eq. (2), according to Basel function it can be obtained $kl^2 = 4.013$ and critical buckling force can be obtained as follows

$$F_C = \frac{4.013\sqrt{BC}}{l^2}\,. \tag{3}$$

3. Plate under bending and torsion moments

In this case the loading consist of bending moment M_x and torsion moment M_z. The solutions of such problem are not found in literature. The following can be written

$$B\frac{d^2u}{dz^2} = -\varphi F z \tag{4}$$

$$C\frac{d\varphi}{dz} = Fz\frac{du}{dz} + M_{tor} \tag{5}$$

After the embedding of $C/F = D$; $M_{tor}/F = H$ in Eq. (4) and Eq. (5), it can be written the following

$$\frac{du}{dz} = D\frac{d\varphi}{dz}z^{-1} - Hz^{-1} \tag{6}$$

$$\frac{d^2u}{dz^2} = D\left(\frac{d^2\varphi}{dz^2}z^{-1} - \frac{d\varphi}{dz}z^{-2} + Hz^{-2}\right) \tag{7}$$

$$BD\left(\frac{d^2\varphi}{dz^2}z^{-1} - \frac{d\varphi}{dz}z^{-2}\right) + Hz^{-2} = -\varphi F z \tag{8}$$

As $F/BD = K$ follows

$$\frac{d^2\varphi}{dz^2}z^{-1} - \frac{d\varphi}{dz}z^{2} + \varphi K z + \frac{H}{D}z^{-2} = 0 \tag{9}$$

The solution of this equation can be obtained by the method of Lagrange constants varying. But some difficulties arise as the buckling conditions are obtained. Therefore the engineering energy method is chosen in this study.

4. The solution of buckling plate under bending moment using energy method

Potential energy U used for lateral beam bending and torsion is calculated as follows

$$U = U_b + U_{tor} \tag{10}$$

where U_b is potential energy of beam lateral bending; U_{tor} is potential energy of beam torsion.

If a section rotates in angle φ, which is the function of length z then potential energy of the beam lateral bending can be calculated as follows

$$U_b = \int_0^l \frac{M_b^2 d^2}{2EI_y} = \int_0^l \frac{F^2\varphi^2 z^2 dz}{2B} = \frac{F^2}{2B}\int_0^l \varphi^2 z^2 dz \qquad (11)$$

The potential energy of torsion can be calculated as

$$U_{tor} = \int_0^l \frac{M_{tor}^2 dz}{2C} \qquad (12)$$

In the case of torsion

$$M = C\frac{d\varphi}{dz} \qquad (13)$$

Therefore

$$U_{tor} = \frac{1}{2}C\int_0^l \left(\frac{d\varphi}{dz}\right)^2 dz \qquad (14)$$

The displacement caused by force F

$$\delta = \frac{\partial U_b}{\partial F} + \frac{\partial U_{tor}}{\partial F} = \frac{F}{B}\int_0^l \varphi^2 z^2 dz \qquad (15)$$

The job

$$W = F\delta = \frac{F^2}{B}\int_0^l \varphi^2 z^2 dz \qquad (16)$$

As $U = W$

$$\frac{F^2}{2B}\int_0^l \varphi^2 z^2 dz + \frac{1}{2}C\int_0^l \left(\frac{d\varphi}{dz}\right)^2 dz = \frac{F^2}{B}\int_0^l \varphi^2 z^2 dz \qquad (17)$$

Then

$$F_{cr}^2 = BC\frac{\int_0^l \left(\frac{d\varphi}{dz}\right)^2 dz}{\int_0^l \varphi^2 z^2 dz} \qquad (18)$$

Noted as follows

$$\varphi = al^2 - az^2$$

$$\int_0^l \left(\frac{d\varphi}{dz}\right)^2 dz = 4\int_0^l (az)^2 dz = \frac{4}{3}a^2 l^3$$

$$\int_0^l \phi^2 z^2 dz = \int_0^l \left(al^2 - az^2\right)^2 z^2 dz =$$

$$= \int_0^l \left(a^2 l^4 z^2 dz - 2a^2 l^2 z^4 + a^2 z^6\right) dz = \frac{8}{105}a^2 l^7$$

$$F_{cr}^2 = BC\frac{\frac{4}{3}a^2 l^3}{\frac{8}{105}a^2 l^7} \qquad (19)$$

Then

$$F_{cr} = 4.18\frac{\sqrt{BC}}{l^2} \qquad (20)$$

If the comparison to exact Timoshenko [1] solution is made, the error of 4% is obtained.

This result for engineering calculations is accepted as the errors defined in standards [6] are up to 5%.

5. The solution of buckling plate under complex loading using energy method

Let torsion moment caused by bending load noted as M_{tor} and additional torsion moment as M_{1tor}. Then the additional torsion angle $\varphi_1 = a_1 l^2 - a_1 z^2$, the work $W_\Sigma = W + W_1$, and potential energy $U_\Sigma = U + U_1$.

Then follows

$$M_{1,tor} = C\frac{d\varphi_1}{dz} \qquad (21)$$

$$U_\Sigma = \frac{F^2}{2B}\int_0^l \phi^2 z^2 dz + \frac{1}{2}C\int_0^l \left(\frac{d\phi}{dz}\right)^2 dz + $$
$$+\frac{1}{2}C\int_0^l \left(\frac{d\phi_1}{dz}\right)^2 dz \qquad (22)$$

There

$$U_1 = \frac{1}{2}C\int_0^l \left(\frac{d\varphi_1}{dz}\right)^2 dz = \frac{1}{2}C\int_0^l M_{1,tor} dz \qquad (23)$$

Additional job as $d\varphi_1 = \frac{M_{1,tor} dz}{C}$

$$W_1 = \int_0^l M_{1,tor} d\varphi_1 = \frac{M_{1,tor}^2 dz}{C} \qquad (24)$$

Then

$$W_\Sigma = \frac{F^2}{B}\int_0^l \varphi^2 z^2 dz + \frac{1}{C}\int_0^l M_{1,tor}^2 dz \qquad (25)$$

As $U_\Sigma = W_\Sigma$, the following can be written

$$\frac{F^2}{2B}\int_0^l \phi^2 z^2 dz + \frac{1}{2}C\int_0^l \left(\frac{d\phi}{dz}\right)^2 dz + \frac{1}{2}\int_0^l \frac{M_{1,tor}^2}{C} dz =$$
$$= \frac{F^2}{B}\int_0^l \phi^2 z^2 dz + \frac{1}{C}\int_0^l M_{1,tor}^2 dz \qquad (26)$$

The Eq. (26) can be written as

$$\frac{1}{2}C\int_0^l \left(\frac{d\phi}{dz}\right)^2 dz = \frac{F^2}{2B}\int_0^l \phi^2 z^2 dz + \frac{1}{2C}\int_0^l M_{1,tor}^2 dz$$

From here

$$\frac{1}{2}C\int_0^l\left(\frac{d\varphi}{dz}\right)^2dz-\frac{M_{1,tor}^2 l}{2C}=\frac{F^2}{2B}\int_0^l\varphi^2z^2dz \quad (27)$$

After noted $\frac{M_{1,tor}^2 l}{C}=S$

$$\frac{F^2}{2B}=T$$

$$\frac{1}{2}C=V$$

Eq. (27) can be written

$$V\int_0^l\left(\frac{d\varphi}{dz}\right)^2dz-T\int_0^l\varphi^2z^2dz-S=0 \quad (28)$$

If $\varphi=al^2-az^2$, then

$$\frac{d\varphi}{dz}=-2az\,;\ \int_0^l\left(\frac{d\varphi}{dz}\right)^2dz=\frac{4a^2l^3}{3}$$

and

$$\int_0^l\varphi^2z^2dz=\frac{8}{105}a^2l^7$$

Eq. (28) can be written as

$$V\frac{4a^2l^3}{3}-T\frac{8}{105}a^2l^7-S=0 \quad (29)$$

or

$$\frac{2Ca^2l^3}{3}-\frac{4F^2a^2l^7}{105B}-\frac{M_{1,tor}^2 l}{2C}=0 \quad (30)$$

In Eq. (30) buckling condition is defined by the ratio of force F to the torsion moment $M_{1,tor}$. The solution of this equation as $M_{1,tor}=0$ will be equal to Eq. (20). This confirms the accuracy of Eq. (30).

One constant of Eq. (30) a is not defined. It can be determined from experiment.

As it is known the displacement can be calculated as follows

$$\delta=\frac{F}{B}\int_0^l\varphi^2z^2dz$$

Then

$$\delta_{cr}=\frac{8}{105B}F_{cr}a^2l^7 \quad (31)$$

$$a=\sqrt{\frac{105\delta_{cr}B}{8F_{cr}l^7}} \quad (32)$$

If the displacement is known for any F_{cr} the calculated constant can be used for the solution of Eq. (30) as loading is bending moment.

Table

Experimental and analytical data

Cross-section dimensions, mm	Length l, m	Force F_{exp}, N	$\sigma_{1,exp}$, MPa	$M_{1,tor}$, Nm (exp. and anal.)	Displacement δ_{cr}, mm	Constant a, $\frac{1}{m^2}$	Force F_{anal}, N	$\sigma_{1,anal}$, MPa	τ, MPa (exp. and anal.)	$\frac{\Delta\sigma_1}{\sigma_{1,exp}}\cdot 100$, %
1	2	3	4	5	6	7	8	9	10	11
50×5	1.0 1.5 2.0	522 230 137	251 166 128	0	-	-	537 238 134	258 172 125	0	+2.8 +3.6 -2.3
	1.0 1.5 2.0	512 218 134	254 158 125	250	1.5 24 189	0.06	528 218 130	262 158 121	32	+3.1 0 -3.2
	1.0 1.5 2.0	483 215 124	240 156 116	500	1.43 23.8 175		501 215 120	249 156 112	64	+3.75 0 -3.4
	1.0 1.5 2.0	447 202 109	222 147 102	1000	1.1 22.5 153		463 202 105	230 147 98	128	+3.6 0 -3.9

Experiments were made for constructional elements strengthened in one end and loaded by transversal force and torsion moment on free end [7].

Experimental and analytical results are presented in Table.

6. Conclusions

1. If the carrying ability of buckling plate is analysed the solutions are needed as the bending moment is used alongside with torsion this. Such problems are not analysed in literature.

2. As the plate is loaded by bending and torsion moments, the solution of buckling equation can be obtained by the method of Lagrange constants varying. But some difficulties arise as the buckling conditions are obtained. The engineering energy method has more acceptability.

3. The buckling solution in the case of bending and additional torsion loading was obtained in this study. It provided good agreement with experimental results and the error did not exceed 4%.

References

1. **Timoshenko, S., Gere, J.** Stability of Elastic System. -New York: McGraw - Hill Company, 1961.-541p.
2. **Wang, C.M., Wang, C.Y., Reddy, J.N.** Exact Solutions for Buckling of Structural Members. Florida: CRC series in Computational Mechanics and Applied Analysis.-CRC PressLLc, Boca Ration, 2005.-245p.
3. **Larul, B., Khell, A., Gueury, M.** Elastic flexural – torsional buckling of steel beams with rigid and continuous lateral restraints.-J. of Constructional steel Research, 2006, p.17.
4. **Chen, W.F., Lui, E.M.** Structural Stability: Theory and Implementation.-New York: Elsevier, 1987.-348p.
5. **Kudzys, A., Kvedaras, A.K.** Research on structural resistance and safety of turbular composite members. -Mechanika. -Kaunas: Technologija, 2006, Nr.5(61), p.5-10.
6. STR 2.05.08.2005. Technical Regulations of Building Construction. Design of Steel Structures. General Attitudes.-Vilnius: Ministry of Environments, 2005.
7. **Žilinskaitė, A., Žiliukas, A.** Stability of steel construction elements in case of bending and torsion. -Proc. of 4th Int. Conf. Strength, Durability and Stability of Materials and Structures (SDSM'04), Palanga. -Kaunas: Technologija, 2007, p.94-99.

A. Žiliukas

SUDĖTINGAI APKRAUNAMŲ PLOKŠTELIŲ KLUPDYMAS

R e z i u m ė

Lenkimo momento apkrovos veikiamos ilgos ir plonos plokštelės dažnai netenka stabilumo. Sprendiniai lenkimo apkrovų poveikiui nustatyti yra gauti, o lenkimo momento ir papildomo sukimo momento poveikiui nustatyti tokių sprendinių nėra. Siūloma stabilumo lygtis gali būti išsprendžiama naudojantis Lagranžo konstantų variacijomis, tačiau sunku nustatyti kraštines sąlygas. Paprasčiau ši problema sprendžiama energiniu metodu. Darbe siūlomas naujas stabilumo lygties sprendinys lenkimo ir papildomo sukimo atveju.

A. Žiliukas

PLATE BUCKLING UNDER COMPLEX LOADING

S u m m a r y

Long and thin cantilever plates loaded by bending moment often lose the stability. The solutions as the bending moment is used are investigated, but as the loading is the bending moment and torsion moment, such problems are not analysed. In this study the buckling equation was obtained and it can be solved by the method of Lagrange constants varying. But mathematical difficulties arise as the buckling conditions are obtained. The engineering energy method has more acceptability. Therefore the new solution of buckling equation was obtained in this study as the loading is the bending moment and torsion moment.

А. Жилюкас

УСТОЙЧИВОСТЬ ПЛАСТИН ПРИ СЛОЖНОМ НАГРУЖЕНИИ

Р е з ю м е

При воздействии изгибающего момента длинные тонкие пластины могут терять устойчивость. Существуют решения для случая воздействия нагрузок, вызывающих изгиб. Однако задачи, когда кроме изгибающего момента, действует дополнительный крутящий момент, практически не рассмотрены. Предлагаемое уравнение устойчивости можно решить методом вариации произвольных постоянных Лагранжа, но при этом затруднительно удовлетворение краевым условиям. Эта задача проще решается энергетическим методом. В работе предложено новое решение уравнения устойчивости для случая изгиба и дополнительного кручения.

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