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An analysis of beam elongation influence to postbuckling displacements under displacement dependent axial force

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1. Introduction

In the analysis of buckling and postbuckling of beams and plates, applied load is commonly considered to be a constant in direction and intensity (i.e. dead-load) [1-3]. In the case of beams, a very small increase in the load over the critical value produces extremely large displacement from the primary equilibrium position [1], so the assumption about the unchanged intensity and direction of the load may be incorrect. Axial deformation of a beam is neglected because of large lateral displacements and constant axial force. In practice, the applied axial force, which causes buckling, often arises from the compression of another elastic part of the system (elastic support, adjacent member in truss, etc.), and this force may depend on the postbuckling displacement [4]. An assumption that treats this load as a dead load is satisfactory for the calculation of the critical load, however, the postbuckling behavior is significantly different [5, 6] and it is not widely researched. In this case, a displacement dependent force may cause beam elongation and axial deformation must be taken into account.

This paper deals with the numerical and experimental analysis of the influence of beam elongation on postbuckling displacements of a beam under axial force produced by the compression with an elastic bar, which is initially compressed while the beam is in the straight-line position. Axial force produced in this way, is constant in direction but its magnitude changes with the lateral displacements of the buckled beam. This setup makes the analysis different from common problems in the postbuckling analysis of beams [2]. Numerical analysis is done using the finite elements method, where the Euler-Bernoulli beam is considered. Numerical results are obtained by the direct solution of equilibrium equation. An experimental verification is performed for the considered problem. The paper presents experimental setup and experimental results for a simple (pinned-pinned) Euler beam. Numerical results are compared with the obtained experimental results.

2. Problem formulation

Let us consider axially loaded beam as shown in the Fig. 1. The load is applied by compression of the spring of stiffness c_0. Axial force is introduced by initial shortening of the spring for length Δl_{0s} while the beam was restrained in the straight line position. When the beam looses stability of straight line position, lateral displacement causes shortening of support distance and, because of that, elongation of the rod and decreasing of the axial force exerted on the beam. Decreasing of the axial force also results in increasing of the beam length, what causes additional lateral displacements of the buckled beam. It is supposed that elastic properties of the system remain linear under this displacement.

Bended shape of the beam and elongation of the elastic rod may be uniquely determined by the lateral displacement $w(s)$ measured in the natural coordinate system [1].

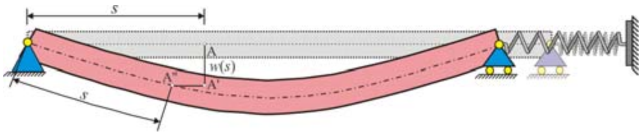

Fig. 1 Axially loaded simple beam for which the load varies during buckling process

3. Numerical analysis

Numerical analysis is performed by using Finite Element Method (FEM). For this purpose, the beam in Fig.1 is divided into n standard Euler beam elements with two nodes and four d.o.f. in the element displacement vector $\{d\}^{e_i} = \{w_1 \; w_{1,s} \; w_2 \; w_{2,s}\}^T$, where w_1 and w_2 are lateral displacements and $w_{1,s}$ and $w_{2,s}$ are derivatives (Fig. 2).

3.1. Interpolation of displacement

Using the derivatives $w_{1,s}$ and $w_{2,s}$ as nodal displacements (which represent sine of slope angles) instead of slope angles θ_1 and θ_2, allows the usage of the same function of interpolation and shape function as in the linear finite elements analysis.

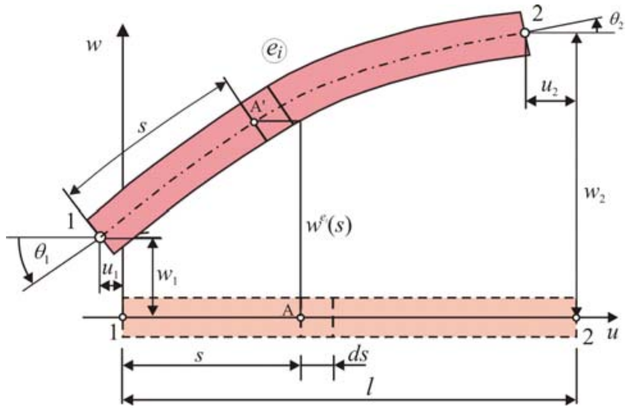

Fig. 2 Deformed shape of the beam finite element in case of large displacements

Considering the standard third order polynomial interpolation function, the displacement of an arbitrary point inside the element e_i could be calculated from the nodal displacements as

$$w^{e_i}(s) = [N]\{d\}^{e_i} = [N_1 \quad N_2 \quad N_3 \quad N_4]\{d\}^{e_i} \tag{1}$$

where N_j, j = 1,...,4 are the coefficients of beam element shape matrix [N].

The shape matrix [N] has the same form as in the case of linear analysis of bending and stability [7]. The difference is only in the coordinate system which is used. Expression (1) holds only in the interior of finite element e_i.

3.2. Potential energy of deformation

Curvature of the beam elastic line in the natural coordinate system be expressed by using the displacement $w^{e_i}(s)$ as [1]

$$\kappa = \frac{w^{e_i}(s)_{,ss}}{\sqrt{1 - w^{e_i}(s)_{,s}^2}} \tag{2}$$

where κ is the curvature and $(\ )_{,s}$ and $(\ )_{,ss}$ are the first and second derivatives with respect to s.

By expanding the expression (2) in the power series, and by considering up to the third order terms, the curvature may be written as

$$\kappa = w^{e_i}(s)_{,ss} + \frac{1}{2} w^{e_i}(s)_{,ss}\, w^{e_i}(s)_{,s}^2 \tag{3}$$

Beam shortening due to lateral displacements (equal to the spring elongation) may be calculated on the basis of displacement $w^{e_i}(s)$ as, [1],

$$\Delta l_b = \sum_{i=1}^{n} \left(l - \int_0^l \sqrt{1 - w^{e_i}(s)_{,s}^2}\, ds \right) \tag{4}$$

where Δl_b is beam shortening and l is length of finite element e_i.

Expanding to power series and limiting to terms of the forth order, the shortening may be written in the following form

$$\Delta l_b = \sum_{i=1}^{n} \int_0^l \left(\frac{1}{2} w^{e_i}(s)_{,s}^2 + \frac{1}{8} w^{e_i}(s)_{,s}^4 \right) ds \tag{5}$$

Deformation energy of the considered system of beam and spring is then

$$U = \frac{1}{2} \sum_{i=1}^{n} B \int_0^l \left(w^{e_i}(s)_{,ss} + \frac{1}{2} w^{e_i}(s)_{,ss}\, w^{e_i}(s)_{,s}^2 \right)^2 ds + \frac{1}{2} c_0 \left(\Delta l_{0s} - \sum_{i=1}^{n} \int_0^l \left(\frac{1}{2} w^{e_i}(s)_{,s}^2 + \frac{1}{8} w^{e_i}(s)_{,s}^4 \right) ds \right)^2 \tag{6}$$

where B is the beam bending stiffness.

Using equation (1) for interpolation of displacements in interior of particular finite element, deformation energy may be written in terms of the nodal displacements as

$$U = \frac{1}{2} \sum_{i=1}^{n} B \int_0^l \Big[\left([N]_{,ss} \{d\}^{e_i} \right)^2 + \left([N]_{,ss} \{d\}^{e_i} \right)^2 \left([N]_{,s} \{d\}^{e_i} \right)^2 \Big] ds + \frac{1}{2} c_0 \left[\Delta l_0 - \sum_{i=1}^{n} \int_0^l \left(\frac{1}{2} \left([N]_{,s} \{d\}^{e_i} \right)^2 + \frac{1}{8} \left([N]_{,s} \{d\}^{e_i} \right)^4 \right) ds \right]^2 \tag{7}$$

3.3. Influence of beam elongation

Decreasing the axial force by $c_0 \Delta l_b$ causes the axial elongation of the compressed beam by

$$\Delta l_{b0} = \frac{c_0 \Delta l_b l_b}{AE} \tag{8}$$

where AE is beam axial stiffness and l_b is beam length.

It is known that small axial shortening of the buckled beam corresponds to high lateral displacement [1]. It is taken into account by defining apparent stiffness of the spring using the equation

$$c_{app} (\Delta l_b + \Delta l_{b0}) = c_0 \Delta l_b \tag{9}$$

Using this value of axial stiffness additional elongation of the spring for Δl_{b0} is simulated.

3.4. Equilibrium equation

Derivation of the expression for deformation energy (7) in sense of displacement get equilibrium equation in the form

$$\sum_{i=1}^{n} \Big[[k]^{e_i} \{d\}^{e_i} + [k_1]^{e_i} \{d\}^{e_i} \Big] - c_{app} \left(\Delta l_{0s} - \sum_{i=1}^{n} \left[\frac{1}{2} \left(\{d\}^{e_i} \right)^T [k_\sigma]^{e_i} \{d\}^{e_i} \right] \right) \times \sum_{i=1}^{n} \left([k_\sigma]^{e_i} \{d\}^{e_i} + [k_{\sigma 1}]^{e_i} \{d\}^{e_i} \right) = 0 \tag{10}$$

where $[k]^{e_i} = \int_0^l B [N]_{,ss}^T [N]_{,ss} ds$ is the element stiffness matrix and $[k_\sigma]^{e_i} = \int_0^l [N]_{,s}^T [N]_{,s} ds$ is the element stress stiffening matrix, adopted from linear analysis [7]. Matrices $[k_1]^{e_i}$ and $[k_{\sigma 1}]^{e_i}$ are nonlinear element stiffness and stress stiffening matrices given by the equations

$$[k_1]^{e_i} = \int_0^l \Big(B \left([N]_{,s} \{d\}^{e_i} \right)^2 [N]_{,ss}^T [N]_{,ss} + B \left([N]_{,ss} \{d\}^{e_i} \right)^2 [N]_{,s}^T [N]_{,s} \Big) ds \tag{11}$$

$$[k_{\sigma 1}]^{e_i} = \int_0^l \Big[\left([N]_{,s} \{d\}^{e_i} \right)^2 [N]_{,s}^T [N]_{,s} \Big] ds \tag{12}$$

Axial load exerted on the beam after buckling is given as

$$P = c_0 \Delta l_{0s} - c_0 \sum_{i=1}^{n} \left[\frac{1}{2} (\{d\}^{e_i})^T [k_\sigma]^{e_i} \{d\}^{e_i} \right] \tag{13}$$

where original spring stiffness c_0 is used.

Summing matrices at the element level over all n finite elements, equilibrium equation (10) may be written in the simplest form

$$[K]\{D\} + [K_1]\{D\} - \\ -P(\{D\})\left([K_\sigma]\{D\} + [K_{\sigma 1}]\{D\}\right) = 0 \tag{14}$$

where $[K]$, $[K_1]$, $[K_\sigma]$ and $[K_{\sigma 1}]$ are global matrices of linear and nonlinear stiffness and stress stiffening, created by standard finite elements procedure from the corresponding element matrices $[k]^{e_i}$, $[k_\sigma]^{e_i}$, $[k_1]^{e_i}$ and $[k_{\sigma 1}]^{e_i}$, and $\{D\}$ is global displacement vector.

3.5. Direct solution of equilibrium equation

The equilibrium equation (14) is a set of nonlinear algebraic equations, and its solution may be obtained by direct numerical solution procedures. In this case the method of linear iterations is used. For this purpose, equation (14) is transformed in the form

$$\{D\} = [K]^{-1}\left(-[K_1]\{D\} + \\ +P(\{D\})\left([K_\sigma]\{D\} + [K_{\sigma 1}]\{D\}\right)\right). \tag{15}$$

Setting initial force $P_0 = c_0 \Delta l_{0s}$ and considering initial solution $\{D\}^{(0)}$, solution of Eq. (14) is determined successively by applying Eq. (15). After solution in i-th iteration is calculated, the solution of $(i+1)$-th iteration is then calculated as

$$\{D\}^{(i+1)} = [K]^{-1}\left(-[K_1]\{D\}^{(i)} + \\ + P(\{D\})^{(i)}\left([K_\sigma]\{D\}^{(i)} + [K_{\sigma 1}]\{D\}^{(i)}\right)\right) \tag{16}$$

Apparent stiffness and axial force in current iteration are calculated as

$$c_{app}^{(i)} (\Delta l_b^{(i-1)} + \Delta l_{b0}^{(i-1)}) = c_0 \Delta l_b^{(i-1)} \tag{17}$$

$$P(\{D\})^{(i)} = P_0 - \frac{1}{2} c_{app}^{(i)} \left(\{D\}^{\mathrm{T}} [K_\sigma] \{D\}\right)^{(i-1)} \tag{18}$$

The calculations are performed until the convergence of displacements is achieved. Solution for the particular simple beam is presented in Fig. 3. It is shown that in case of such kind of axial force moderate displacement of order beam thickness appear.

The convergence of the middle point displacement of a simple beam in sense of required mesh density for different values of initial axial force is shown in Fig. 4. It can be seen that the value of displacement converges for very small number of finite elements.

The number of required iterations for fixed number of finite elements and initial value of axial force is presented in Fig. 5. It can be seen that the number of iteration is almost independent on the number of finite elements, and the result converges after approximately 30 iterations.

The influence of beam elongation on postbuckling displacement of the middle point of simple beam for different axial stiffness of spring is presented in Fig. 6. If the beam elongation is not taken into account, error increases with increasing of axial stiffness of the spring and tends to 30%.

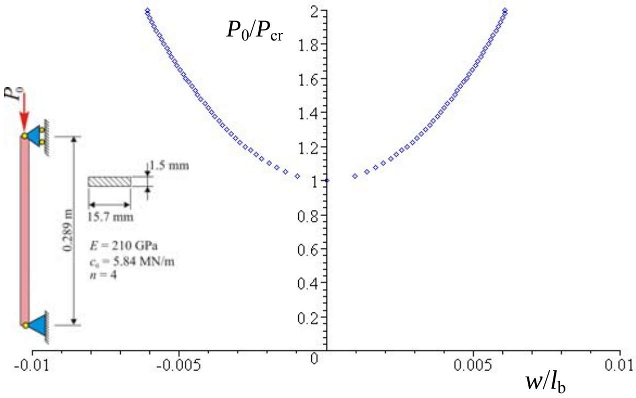

Fig. 3 Postbuckling displacement of middle point of simple beam calculated by linear iteration. Initial solution is first buckling eigenvector

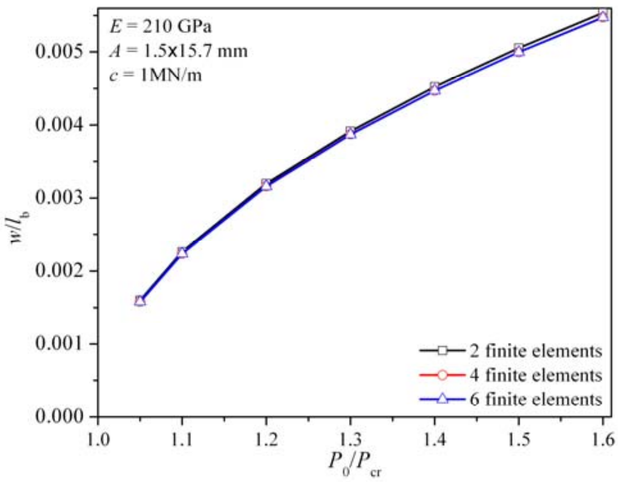

Fig. 4 Value of vertical displacement of the beam middle point for different values of the initial axial force and different number of finite elements

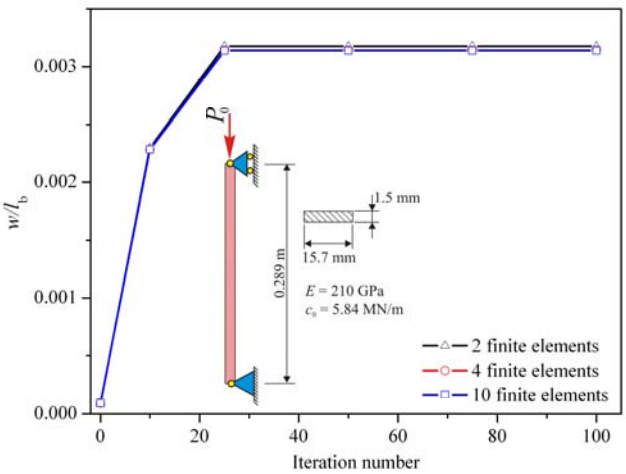

Fig. 5 Convergence of the calculation of the simple beam midpoint displacement. Initial displacement is the buckling eigenvector, where the displacement of midpoint is scaled to w = 0.000092 [m]

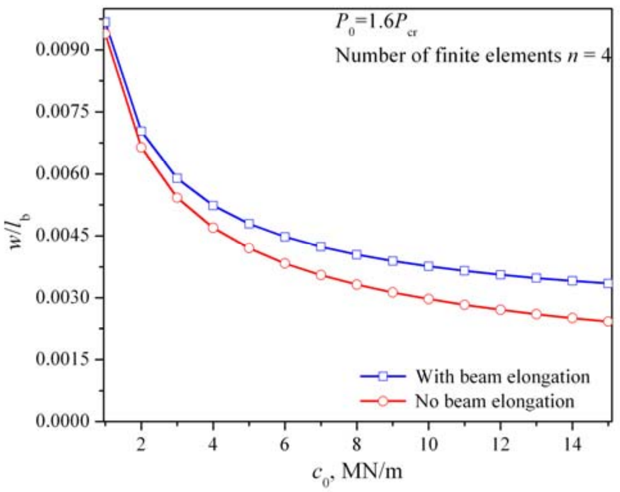

Fig. 6 Difference in the displacement of middle-point of a simple beam if the beam elongation is and it is not taken into account

4. Experimental analysis

An experimental testing using equipment illustrated in Fig. 7 is provided.

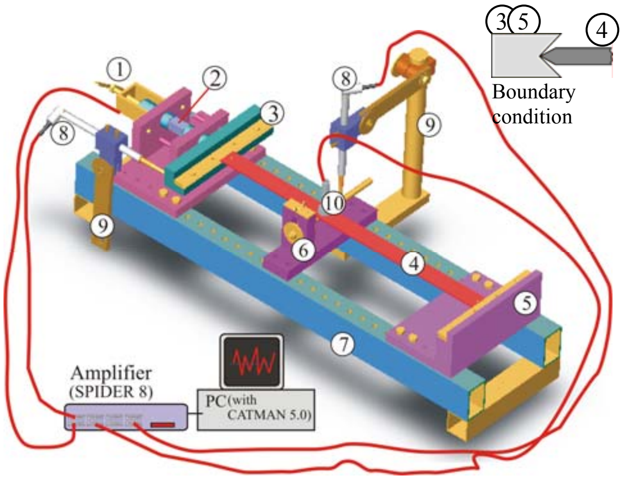

Fig. 7 Schematic view of experimental setup

Beam-like specimen *4* is connected to the testing platform *7* with two supports *3* and *5*. Axial force over axially movable support *3* and measured with strain gage dynamometer *2* is applied. The dynamometer is a hollow circular steel tube with bonded strain gages LY6/120 connected in a full Wheatstone bridge. Dynamometer is also the main bearer of axially movable part of movable support. Displacement is measured with inductive displacement transducer WA20, positioned at the middle point of the beam and on the axially movable support. The beam is restrained in a straight position until the force is regulated by adjusting screw *1* to a desired value $P_0 > P_{cr}$. After initial compression beam is allowed to buckle. Force and displacement of the characteristic point are measured by amplifier SPIDER 8 in transition to the new equilibrium position. This testing for different initial values of the axial force is repeated. To remove the influence of contact force of the displacement transducer, the displacement is measured using the movable rod *6*, which is driven by a screw to make a contact with the buckled beam. Contact between them is registered by the indicator closed current circuit.

The experiment is performed for the prismatic steel beam with the following characteristics: modulus of elasticity E = 207 GPa, length l_b = 0.382 m, and cross sectional area A = 14.3×2.628 mm, critical force P_{cr} = = 302.9 N. The beam is supported by pinned supports, as shown on Fig. 7.

4.1. Identification of axial stiffness

Movable support is a complex system, and axial stiffness of axially movable support is determined experimentally. The change of axial force and axial displacement of the movable support caused by buckling is measured. Fig. 8 presents the change of value of axial force during initial compression and buckling process. During initial compression, force–displacement dependence slightly differs from straight-line because of support adjusting. Large displacements of the beam cause beam shortening, and this shortening also causes decompression of the dynamometer, what is presented by the left part of Fig. 8. The beam snaps to a new position and the sudden change of the axial force appears, what produce vibrations around new buckled configuration. The dependency force-displacement during buckling is linear and defines axial stiffness. Measurement for values of $P_{cr} < P_0 \leq 2P_{cr}$ is performed.

Measured value of axial rigidity of elastic parts in the system of the movable support is c = 14.48 MN/m.

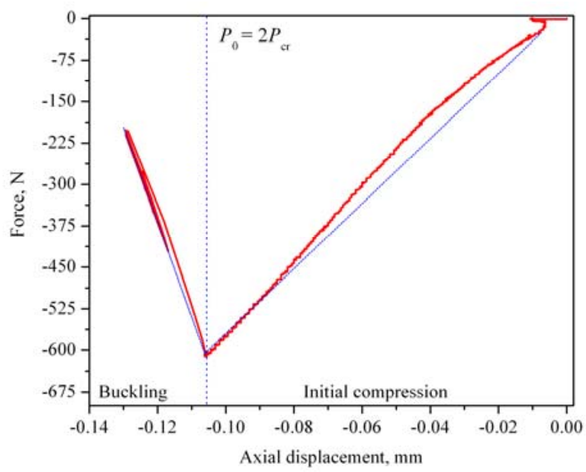

Fig. 8 Numerical and experimental displacements of the beam middle point

5. Comparison of numerical and experimental results

The results of numerical and experimental analysis for the pinned-pinned support condition of a beam are shown in the Fig. 9. Numerical results predict stable postbuckling state, however, due to the existing imperfections in this experiment, the beam tends to buckle on the one side. Buckling in the opposite direction by additional external influence may be achieved. Also, imperfection causes lateral displacements for the value of the axial force that is less than the exact value of the critical force.

Experimental analysis shows that numerical analysis with introduced beam elongation may be used to predict the value of postbuckling displacements. Experimental results depart somewhat from the numerical analysis, because of the imperfections of real beam-like probe and approximations used in the numerical model. Discrepancy of the results slightly increases with the increase of initial value of the axial force.

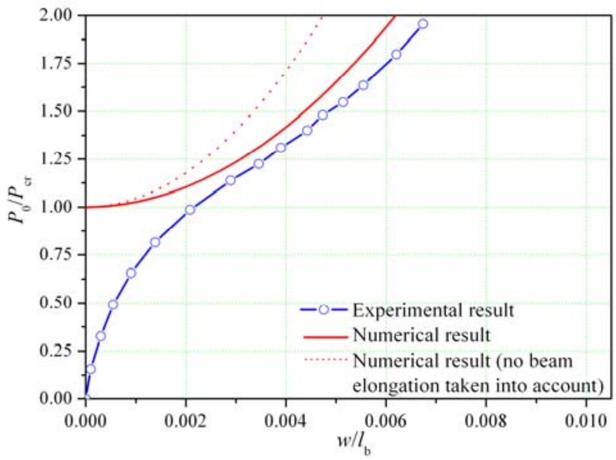

Fig. 9 Comparison of numerical and experimental results for the displacement of middle point of simple beam

6. Conclusion

An influence of beam elongation on equilibrium states of a buckled beam compressed by an elastic body is analysed both numerically and experimentally. In the numerical analysis, the equilibrium equation, containing third order nonlinearities, is derived using the finite elements method. Using the method of linear iterations for different initial axial compression the equilibrium equation is solved. The results already converge under a small number of finite elements, within only a few iterations. It is shown that beam elongation affects the value of lateral displacements, and its influence is increasing with increasing of axial stiffness of body compressing the beam, and may have the value of even 30%. Effect of beam elongation is taken into account by defining apparent axial stiffness of the compressive body.

The experimental results show good coincidence with numerical results for initial value of axial force greater than critical when the beam elongation is taken into account. Displacements of the compressed beam exist when the force has a less value than critical. Accuracy of numerical results decreases for initial axial force close to critical, due to the presence of imperfections in a real system.

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STRYPO PAILGĖJIMO ĮTAKOS KLUPDYMO VIRŠ KRITINĖS RIBOS POSLINKIAMS PRIKLAUSOMAI NUO AŠINĖS JĖGOS POVEIKIO ANALIZĖ

R e z i u m ė

Straipsnyje analizuojama strypo, veikiamo ašine spaudimo jėga, priklausoma nuo deformacijos dydžio, pailgėjimo įtaka klupdymo virš kritinės ribos poslinkiams. Atlikta skaitinė ir eksperimentinė šio poveikio analizė. Skaitinė analizė atlikta baigtinių elementų metodu. Taikant iteracijų metodą sudaryta ir išspręsta netiesinė sistemos pusiausvyros lygtis. Skaitinės analizės rezultatai patikrinti eksperimentiškai. Skaitiniai ir eksperimentiniai tyrimų rezultatai rodo, kad strypo pailgėjimas turi didelę įtaką virš kritinės ribos klupdomo strypo poslinkiams.

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AN ANALYSIS OF BEAM ELONGATION INFLUENCE TO POSTBUCKLING DISPLACEMENTS UNDER DISPLACEMENT DEPENDENT AXIAL FORCE

S u m m a r y

This paper presents an analysis of beam elongation influence on the postbuckling displacements in case of axial compression by a force depending on axial deformation of the beam. It is performed both numerical and experimental analysis of this effect. Numerical analysis is done by using finite elements method. Nonlinear equilibrium equation is derived and solved using the method of simultaneous iterations. Verification of numerical results is done by experimental analysis. Both numerical and experimental results show significant influence of the beam elongation on postbuckling displacements.

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АНАЛИЗ ВЛИЯНИЯ УДЛИНЕНИЯ СТЕРЖНЯ ПРИ ЕГО ВОЗДЕЙСТВИИ ОСЕВОЙ СИЛОЙ, ЗАВИСЯЩЕЙ ОТ ДЕФОРМАЦИЙ НА ЕГО СДВИГ ПРИ СЖАТИИ В ЗОНЕ, ПРЕВЫШАЮЩЕЙ КРИТИЧЕСКИЙ ПРЕДЕЛ

Р е з ю м е

В статье анализируется влияние удлинения стержня на его устойчивость при сжатии осевой силой, зависящей от величины деформации в зоне, превышающей критический предел. Проведен численный и экспериментальный анализ этого эффекта. Численный анализ проведен при использовании метода конечных элементов. Составлена методом мгновенных итераций и решена система нелинейных уравнений равновесия. Результаты числовых исследований проверены экспериментально. Числовые и экспериментальные исследования показывают существенное влияние удлинения стержня на его деформации сжатия в зоне, превышающей критический предел.

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