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Theoretical considerations regarding the virtual modelling of surfaces used in topography

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1. Introduction. Mathematical aspects for determining trajectories

Technology frequently appeals for tracking or tracing of certain curves/paths to their vectorial or parametric representation in the space [1]. If the trajectory, which must be travelled, is given by discrete points (loops), determined by measuring, one possibility is by interpolation of spline curves. These curves are formed from cubic spline arches, given as parametric form, function of one parameter $u \in (0,1)$, connected between them. The connection is required for the first and the second derivate, and the tangents directions to the extremes A and B points of curve are considered to be known.

Noting with r_i, radius vector in loops, connection of two spline adjacent arches ΔC_{i-1} and ΔC_i, (Fig. 1), the vectorial equation of ΔC_i arc has the following form:

$$\vec{r}_i(u) = x_i(u)i + y_i(u)j + z_i(u)k, \quad u \in (0,1) \tag{1}$$

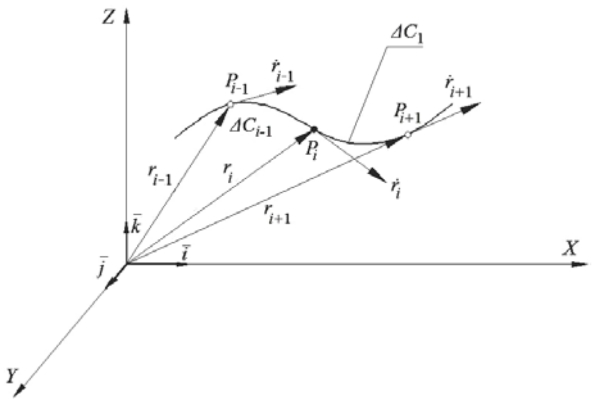

Fig. 1 Continuity of derivates in loops of one cubic spline curve

It shows an interpolation method. The method is based on the *"natural spline"* variant [2].

Thus, we refer to a spline plane curve defined in this way. A surface represented by the measured discrete points can be interpolated as well, considering a number of $n+1$ given points (loops), notated with P_i, (Fig. 2), where $i = 0,...,n-1$.

Considering a reunion of polynomial arches s_i, grade 3, connected and contained between these points, they will form a cubic plane spline curve c_i, grade 3.

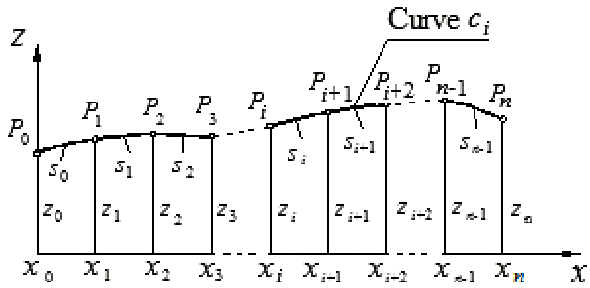

Fig. 2 Curve spline cubic plane

To draw the curve c_i it is necessary to preliminarily determine the arches of curves grade 3, $s_i(x)$, $i = 0,...,n-1$, connected between them, in such a manner to keep the following conditions [3]:

- curve $s_i(x)$ pass through points P_{i-1}, P_i, where $i = 1,...,n$;
- curves $s_i(x)$ and $s_{i+1}(x)$ admit a common tangent in point P_{i+1} where $i = 0,...,n-1$;
- curves $s_i(x)$ and $s_{i+1}(x)$ have a common curvature in the point P_{i+1}, where $i = 0,...,n-1$.

The curves s_i are expressed with the aid of the following type of relation

$$z_i(x) = A_i x^3 + B_i x^2 + C_i x + D_i \tag{2}$$

where the coefficients are real numbers, which must be calculated based on the dates and conditions imposed.

To determine these coefficients, the derivates grade 2, notated with M_i, $i = 0,...,n-1$, derivates which have a linear variation in scale range $x_i - x_{i+1}$, [4], (Fig. 3) are assumed to be unknown in loops.

Thus, we can write

$$tg\alpha = \frac{M_{i+1} - M_i}{h_x} = \frac{z'' - M_i}{x - x_i} \tag{3}$$

$$z'' = M_i \frac{x_{(i+1)} - x}{h_x} + M_{i+1} \frac{x - x_i}{h_x} \tag{4}$$

The function $z''(x)$, expressed by the Eq. (4), which is continuous for $i = 0,...,n-1$, can be integrated twice:

– for the first integration

$$z' = -M_i \frac{(x_{i+1}-x)^2}{2h_x} + M_{i+1}\frac{(x-x_i)^2}{2h_x} + C_1 \tag{5}$$

– for the second integration

$$z = M_i \frac{(x_{i+1}-x)^3}{6h_x} + M_{i+1}\frac{(x-x_i)^3}{6h_x} + C_1 x + C_2 \tag{6}$$

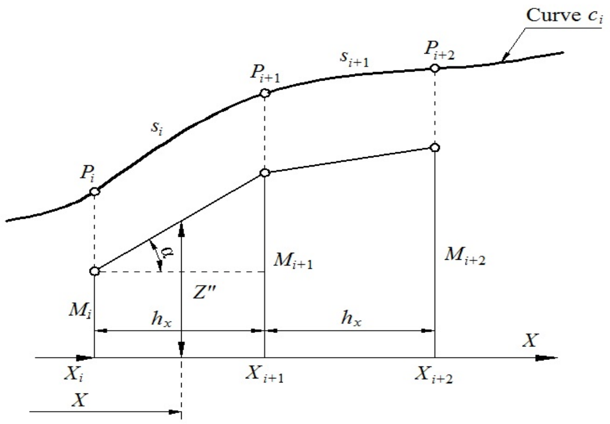

Fig. 3 Approximation of P_i nodes by M_i, second order derivatives

Integration constants C_1 and C_2 from relations (5) and (6) are determined by enforcing condition as for $x = x_i$ and $x = x_{i+1}$ the space curve $s_i(x)$ to go through points P_i and P_{i+1}, it means that, the z coordinate is to take values z_i and z_{i+1}:

$$C_1 = \frac{z_{i+1}-z_i}{h_x} - \frac{M_{i+1}-M_i}{6} h_x \tag{7}$$

$$C_2 = \frac{h_x}{6}\left(M_{i+1}x_i - M_i x_{i+1}\right) + \frac{1}{h_x}\left(x_{i+1} z_i - x_i z_{i+1}\right) \tag{8}$$

Equalizing now the derivate, given by relation (5), at the right side of the arc of curve $s_i(x)$, namely in the point, P_{i+1} of abscise x_{i+1}, with the left derivate of the arc of curve $s_{i+1}(x)$, namely in the same point P_{i+1}, according with relation (6), the following is obtained

$$\begin{aligned} &M_{i+1}\frac{h_x}{2} + \frac{z_{i+1}-z_i}{h_x} - \frac{M_{i+1}-M_i}{6}h_x = \\ &= -M_{i+1}\frac{h_x}{2} + \frac{z_{i+2}-z_{i+1}}{h_x} - \frac{M_{i+2}-M_{i+1}}{6}h_x \end{aligned} \tag{9}$$

Ordering the terms from the relation above, we obtain

$$\begin{aligned} &\frac{h_x}{6}M_i + \frac{2h_x}{3}M_{i+1} + \frac{h_x}{6}M_{i+2} = \\ &= \frac{1}{h_x}\left[\left(z_{i+2}-z_{i+1}\right)-\left(z_{i+1}-z_i\right)\right] \end{aligned} \tag{10}$$

where $i = 0,...,n-2$.

With the aid of relation (11) $n-1$ linear equations expressed can be written, resulting a system of linear equations [5]

$$\begin{pmatrix} \frac{2}{3} & \frac{1}{6} & 0 & 0\ldots & 0 & 0 & 0 & 0 \\ \frac{1}{6} & \frac{2}{3} & \frac{1}{6} & 0\ldots & 0 & 0 & 0 & 0 \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0\ldots & \frac{1}{6} & \frac{2}{3} & \frac{1}{6} & 0 \\ 0 & 0 & 0 & 0\ldots & 0 & \frac{1}{6} & \frac{2}{3} & \frac{1}{6} \\ 0 & 0 & 0 & 0\ldots & 0 & 0 & \frac{1}{6} & \frac{2}{3} \end{pmatrix} \cdot \begin{pmatrix} M_1 \\ M_2 \\ M_3 \\ . \\ . \\ . \\ M_{n-3} \\ M_{n-2} \\ M_{n-1} \end{pmatrix} \cdot \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ . \\ . \\ . \\ c_{n-4} \\ c_{n-3} \\ c_{n-2} \end{pmatrix} \tag{11}$$

where $c_i = \frac{1}{h_x^2}\left[\left(z_{i+2}-z_{i+1}\right)-\left(z_{i+1}-z_i\right)\right]$.

For solving the linear system with $n+1$ unknowns M_i, two more limitrophe conditions are necessary. These conditions as presented, are referring to the fact that the grade 2 derivates (curvatures), in the extremity points, namely in P_0 and P_n, to be null, meaning that, $M_0 = 0$ and $M_n = 0$ [6].

After calculating the unknowns M_i, return to the equation (2), replace the values of C_1 and C_2 constants calculated with equations (7) and (8), and thus for the arc of curve $s_i(x)$ the following equation is obtained

$$\begin{aligned} z_i(x) &= M_i \frac{(x_{i+n}-x)^3}{6h_x} + M_{i+n}\frac{(x-x_i)^3}{6h_x} + \\ &+ \left(z_i - \frac{M_i h_x^2}{6}\right)\frac{x_{i+n}-x}{h_x} + \left(z_{i+n} - \frac{M_{i+n}\cdot h_x^2}{6}\right)\left(\frac{x-x_i}{h_x}\right) \end{aligned} \tag{12}$$

Thus, the coefficients A_i, B_i, C_i and D_i of relation (12) are determined with the aid of the next set of expressions:

$$\left.\begin{aligned} A_i &= \frac{1}{6h_x}\left(M_{i+1}-M_i\right) \\ B_i &= \frac{1}{2h_x}\left(M_i x_{i+1} - M_{i+1}x_i\right) \\ C_i &= \frac{M_{i+1}x_i^2 - M_i x_{i+1}^2}{2h_x} + \frac{z_{i+1}-z_i}{h_x} + \frac{\left(M_i - M_{i+1}\right)h_x}{6} \\ D_i &= \frac{1}{6h_x}\left(M_i x_{i+1}^3 - M_{i+1}x_i^3\right) + \\ &+ \frac{h_x}{6}\left(M x_i - M_i x_{i+1}\right) + \frac{1}{h_x}\left(x_{i+1} z_i - x_i z_{i+1}\right) \end{aligned}\right\} \tag{13}$$

note that, x represents the variable in the $x_i - x_{i+1}$ range.

2. Determination of on surface defined by the network points

Choosing a second plane spline curve $\gamma_i = Z_i(y)$ perpendicular on the plane of $c_i = Z_i(x)$ and having common loops so that the arc of curve projections $s_i(y)$, which belong to the curve γ_i, on the plane $z = 0$ (level plan) must be also equidistant from the length h_y, a rectangular spatial equidistant network with common nodes can be formed, respectively a spline bicubic surface (Fig. 5), [6 - 8].

The spline curves c_i parallel to the OXZ plane will be transversal sections, and the curves γ_i parallel to the OYZ plane, longitudinal sections, (Figs. 4, 5). These curves also represent the coordinate lines of Coons surface.

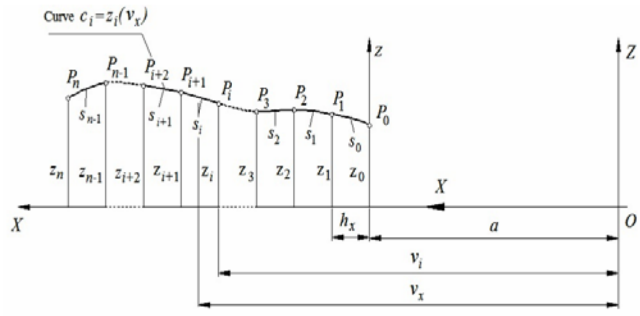

Fig. 4 Changing the reference system

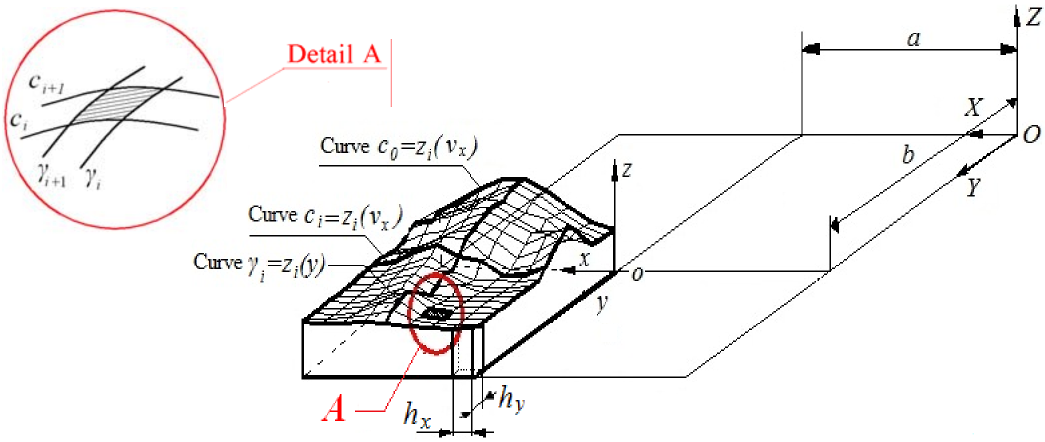

Fig. 5 The bicubic spline surface

The final form of the equation of the arc of curve belonging to the plane spline curve c_i, is

$$z_i(\upsilon) = A_i \cdot (\upsilon - \upsilon_i)^3 + B_i \cdot (\upsilon - \upsilon_i)^2 + C_i \cdot \upsilon + E_i \qquad (14)$$

for $\upsilon_i < \upsilon < \upsilon_{i+1}$.

The equation of curve $c_i = z_i(\upsilon)$ will consist of a set of $n-1$ relations like the one above.

3. Applications in topography and cadastre area

In the field of topography and cadastre a major problem is represented by the virtual and real modelling of the surfaces (relieves).

Thus, for solving this problem it has been started from the approximation of the surfaces using bicubic spline curves, which were mathematically determined the fact which allowed virtual representation of the surface with the aid of the Borland Delphi 7 software.

The application has been developed starting with the topographic map of mountain Vladeasa [9], according to figures 6, respectively 7, provided with level curves, which represent constant heights by level 0, sea level, as an example, the 1050 or 1200 curves delimit a four-square area, (Fig. 6).

Then, an approximation of level curves through the points is made, with the aid of a graphic program named ArchiCAD, thereby obtaining a set of points (nodes) in three coordinates.

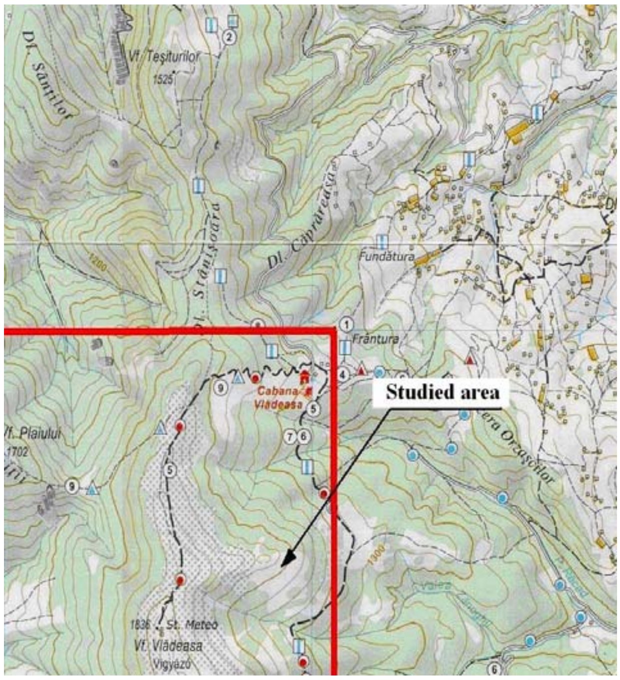

Fig. 6 Topographic map of mountain Vladeasa

With the help of a new topographic program, the nodes are ordered into a rectangular system, in this case of $9 \times 20 = 180$ points. At the scale of 1:50000, the values are listed in centimetres, and the achieved network is represented graphically like in Fig. 5, with the specification that in the given case $a = 12{,}7845$ mm, $b = 17{,}339472$ mm,

$h_x = 0{,}79245283$ mm and $h_y = 0{,}65656566$ mm.

For a more accurate representation of the Coons surface defined by the $9 \times 20 = 180$ nodes, the number of these has grown at $65 \times 58 = 3770$. In this case the new element of Coons surface will be delimitated by the heights $h'_x = 0{,}099056603$ mm and $h'_y = 0{,}21885522$ mm, achieved with the aid of relations:

$$h'_x = \frac{0.79245283}{8} \qquad h'_y = \frac{0.65656566}{3} \tag{15}$$

The equations of $65 \times 58 = 3770$ cubic spline arches were calculated dependent to the υ parameter, in relation to the OXYZ system, accordingly to relations (15).

For a fast and efficient calculation of coefficients A_i, B_i, C_i, D_i and E_i knowing the values for x_i, z_i, M_i where $i = 0, \ldots, n-1$, a logical diagram and a program in programming language Borland Pascal 7.0 have been created, which allow the settlement of a batch of relations for every curve $c_i = z_i(\upsilon)$.

Equations example for transversal curve

$$c_0 = \begin{cases} 0.94416 + 0.12286\nu + 0.099214(\nu - 12.688)^3 & \nu < 13.480 \\ -1.526 + 0.30978\nu + 0.23587(\nu - 13.480)^2 - 0.11075(\nu - 13.480)^3 & \nu < 14.273 \\ -3.7908 + 0.47496\nu - 0.027419(\nu - 14.273)^2 + 0.0010525(\nu - 14.273)^3 & \nu < 15.065 \\ -3.1827 + 0.43349\nu - 0.024917(\nu - 15.065)^2 - 0.12271(\nu - 15.065)^3 & \nu < 15.858 \\ 1.0329 + 0.16281\nu - 0.31665(\nu - 15.858)^2 - 0.12271(\nu - 15.858)^3 & \nu < 16.650 \\ 6.5198 - 0.17608\nu - 0.11101(\nu - 16.650)^2 + 0.00050454(\nu - 16.650)^3 & \nu < 17.443 \\ 9.5351 - 0.35297\nu - 0.11221(\nu - 17.443)^2 + 0.076557(\nu - 17.443)^3 & \nu < 18.235 \\ 10.116 - 0.38657\nu + 0.069798(\nu - 18.235)^2 - 0.029360(\nu - 18.235)^3 & \end{cases}$$

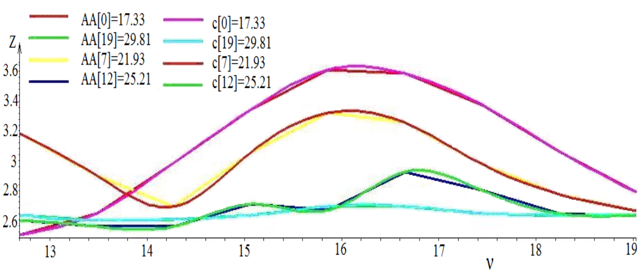

Fig. 7 Transversal sections

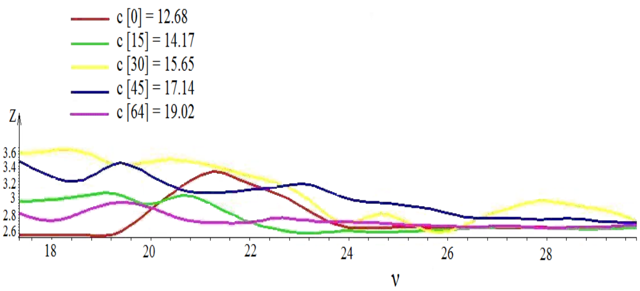

Fig. 8 Longitudinal sections

The transversal sections are represented in Fig. 7 and the longitudinal sections are illustrated in Fig. 8.

Equations example for longitudinal curve

$$CJ_0 = \begin{cases} 2.5648 - 0.0035647\nu + 0.0047361(\nu - 17.339)^3 & \nu < 17.996 \\ 2.4559 + 0.0025601\nu + 0.0093286(\nu - 17.996)^2 - 0.034280(\nu - 17.996)^3 & \nu < 18.653 \\ 3.0487 + 0.029522\nu - 0.058192(\nu - 18.653)^2 + 0.27018(\nu - 18.653)^3 & \nu < 19.309 \\ -2.1711 + 0.24346\nu + 0.47397(\nu - 19.309)^2 - 0.26205(\nu - 19.309)^3 & \nu < 19.966 \\ -7.7011 + 0.52696\nu - 0.042193(\nu - 19.966)^2 - 0.020468(\nu - 19.966)^3 & \nu < 20.622 \\ -6.0366 + 0.44508\nu - 0.082510(\nu - 20.622)^2 - 0.25671(\nu - 20.622)^3 & \nu < 21.279 \\ 3.2249 + 0.0047521\nu - 0.58814(\nu - 21.279)^2 - 0.38659(\nu - 21.279)^3 & \nu < 21.935 \\ 9.0551 - 0.26761\nu + 0.17332(\nu - 21.935)^2 - 0.38729(\nu - 21.935)^3 & \nu < 22.592 \\ 9.4073 - 0.28224\nu - 0.19559(\nu - 22.592)^2 + 0.12939(\nu - 22.592)^3 & \nu < 23.249 \\ 11.443 - 0.37174\nu + 0.059275(\nu - 23.249)^2 + 0.11136(\nu - 23.249)^3 & \nu < 23.905 \\ 6.1941 - 0.14989\nu + 0.27862(\nu - 23.905)^2 - 0.14379(\nu - 23.905)^3 & \nu < 24.562 \\ 1.8541 + 0.030026\nu - 0.0045986(\nu - 24.562)^2 - 0.020252(\nu - 24.562)^3 & \nu < 25.218 \\ 2.6595 - 0.0022027\nu - 0.044489(\nu - 25.218)^2 + 0.037538(\nu - 25.218)^3 & \nu < 25.875 \\ 2.9065 - 0.012077\nu - 0.029449(\nu - 25.875)^2 - 0.016837(\nu - 25.875)^3 & \nu < 26.531 \\ 2.4661 + 0.0048189\nu - 0.0037153(\nu - 26.531)^2 + 0.0050795(\nu - 26.531)^3 & \nu < 27.188 \\ 2.4200 + 0.0065093\nu - 0.0062898(\nu - 27.188)^2 - 0.014380(\nu - 27.188)^3 & \nu < 27.845 \\ 2.6958 - 0.0034405\nu - 0.021444(\nu - 27.845)^2 + 0.012377(\nu - 27.845)^3 & \nu < 28.501 \\ 3.0364 - 0.015593\nu + 0.0029347(\nu - 28.501)^2 + 0.035236(\nu - 28.501)^3 & \nu < 29.158 \\ 1.6066 + 0.033829\nu + 0.072339(\nu - 29.158)^2 - 0.036726(\nu - 29.158)^3 & \end{cases}$$

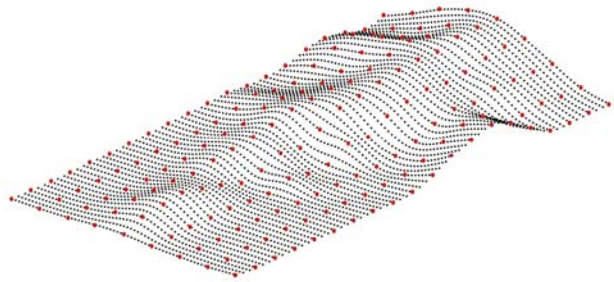

Fig. 9 The discretized surface

The entire discretized surface has been presented axonometrically in Fig. 9, with the specification that, the given network of nodes has been marked in red.

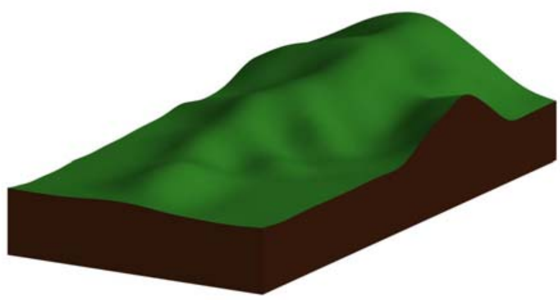

Fig. 10 The real surface

So, the finished network has been coloured and filled with Coons surface elements and a real surface represented in the above figure has been achieved, (Fig. 10).

4. Conclusions

In the domains of topography and cadastre, a major problem is represented by the virtual and real modelling of surfaces (relieves). For this purpose, it started from the approximation of the surfaces through bicubic spline curves, which were mathematically determined, the fact that allowed virtual representation of the surface with the aid of the Borland Delphi 7 software.

A real situation by delimitating a certain surface from the topographic map illustrated in Fig. 6, was considered. Then, an approximation of level curves through the points is made with the aid of a graphic program named ArchiCAD, thereby obtaining a set of points (nodes) in three coordinates.

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TEORINĖS PASTABOS APIE EFEKTYVŲ PAVIRŠIŲ MODELIAVIMĄ, NAUDOJAMĄ TOPOGRAFIJOJE

R e z i u m ė

Straipsnyje pristatomas galimas paviršiaus modeliavimo pritaikymas topografijos ir kadastro srityse, kur pagrindinė problema yra virtualus modelis ir realūs paviršiai (reljefai).

Šiam tikslui yra sudarytas matematinis modelis, loginė schema ir programa, leidžianti nustatyti duotus taškus/mazgus, panaudojant bikubines sklandžias kreives (trajektorijas) leidžiančias aproksimuoti paviršius.

Šis metodas taikytas braižant Vlediasos kalno topografinį žemėlapį paviršiaus atraminiams taškams nustatyti pagal lygio kreives , gautas naudojant taškus (mazgus) trijų koordinačių sistemoje.

Nustatytos per šiuos taškus (mazgus) einančios kreivės suderinančios taškų tinklą, kuris yra užbaigtas spalvotais Kunso (Coons) paviršiaus elementais, ir taip gautas apribotas tikrasis paviršius.

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THEORETICAL CONSIDERATIONS REGARDING THE VIRTUAL MODELLING OF SURFACES USED IN TOPOGRAPHY

S u m m a r y

This paper presents a possible application in the field of topography and cadastre, where a major problem is to virtual model real surfaces (relieves).

For that, a mathematical model, logical diagram and an program which allow to determine the given points/nodes through curves/pathes were developed. The bicubic spline curves which allowed to approximate surfaces were considered.

The application is developed starting from the topographic map of Vladeasa mountain by delimiting of an area to which an approximation of level curves through points (nodes) in three coordinates is made.

The curves which pass through these points (nodes) are determined obtaining a network of points which are completed in colours for Coons surface elements, thus achieving a bounded virtual surface.

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ТЕОРЕТИЧЕСКИЕ ЗАМЕЧАНИЯ ОБ ЭФФЕКТИВНОМ МОДЕЛИРОВАНИИ ПОВЕРХНОСТЕЙ В ТОПОГРАФИИ

Р е з ю м е

Статья представляет возможное применение моделирования в топографии и кадастре, где основной проблемой является виртуальная модель и реальные поверхности (рельефы).

Для этой цели составлена математическая модель, логическая схема и программа, позволяющая определить данные точки/узлы при использовании кривых/траекторий. Это определяет кривые бикубического сплайна позволяющие аппроксимировать поверхности.

Применение метода осуществлено используя топографическую карту горы Владеаса при определении опорных точек поверхности, для которой составлены кривые уровня используя точки/узлы в трехмерной системе координат.

Определены кривые, проходящие через эти точки/узлы, образующие сеть точек, которая заканчивается цветными элементами поверхности Кунса (Coons), так определяя ограниченную действительную поверхность.

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