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Lie Group Variational Integrator for Multi-body System with Rotation Coupling in Space

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1. Introduction

The rotation coupling in space lead to many difficulties to dynamics modeling for multi-body system. Firstly, the rotation coupling in space make kinematics relation with strong nonlinear character. Secondly, the triangle expression of rotation makes the kinematics and dynamic equation with long and complex expression, which is not suit to the programming realization and difficulty to be solved. The classical multi-body dynamics modeling methods base on topology relation and Newton mechanics, which are convenient for the multibody system in planar, but for a multibody system in space, the attitude transformation and vector relation become abstract. So, the method which don't need space analysis seems more suit to complex system dynamics modeling, like the Lagrange or Hamilton methods.

The computational geometry mechanics method which is represented by Lie group variational integrator offers a new way to the multi-body dynamics modeling. With the Lie group and Lie algebra theory, the relationship between rotation angle and attitude is established by the exponential mapping. The dynamic equation is obtained by the variation to attitude matrix, which avoid the complex triangle transformation. With Legendre transformation, the Hamilton equation use momentum as parameter, so the differential calculation to it is also avoided. The dynamics equation is simplified further.

In recent years, the main results of geometry mechanics exploration are as follows. Ding [1] explored the implicit solution method for the multibody dynamics system with Lie group theory, and the constraint violating problem is avoided. Chen [2] explored the multi-symplectic Lie group variational integrator of flexible multibody system, the dynamics model of flexible body is built on SE(3). Celledoni [3] explored the Lie group variational integrator of multi-stage spherical pendulum in space, and the geometry dynamics modeling and motion planning method for UAV is also researched. Muller [4] explored the Lie group structure of time derivatives of equations of motion and the compact equation which is easily to be parameterized is obtained. Li [5] explored the friction contact problem, which is expressed as a horizontal linearity problem and implant into the Lie group variational integrator structure. Paz [6] used the Lie algebra and multi linear operators to recursively explore the sparsity characters in linearization problem. Muller [7] obtained the kinematics and dynamics model of parallel mechanism with Lie group method, and the invariant frame of Lie group is used for modular modeling. Hong [8] built the geometry dynamic model of multirotor aerial vehicle with Lie group method, which include all masses, inertias, rotor thrust forces and moments and external aerodynamic of the MAV body and rotors. Muller [9] proofed the right-trivialized differential of the exponential and Cayley map and their directional derivatives, and a generalized-alpha scheme for rigid/ flexible multibody systems in terms of the Cayley map with improved computational efficiency is also derived out. Bai [10] built four types of Lagrange equations and four types Hamilton equations of double pendulum in space, and the computation characters in long time simulation is compared. Tang [11] proposed a modified extended Lie-group differential algebraic equation method for solving index-3 Hessenberg-DAEs which exhibits a competitive performance in terms of high accuracy and the preservation of algebraic constraints. Fang [12] verified the half-implicit Lie integrator allows a more straightforward formulation of DAEs and the Jacobians and leads to faster convergence in friction and contact problem. Rousso [13] built the kinematics and dynamics equation of space manipulator on the Special Euclidean group SE(3), and the input-output linearization of the system is performed on the Lie algebra se(3). Holzinger [14] used Lie group time integration methods to compute consistent updates for the rotation vector or the Euler angles in each time step, the accuracy is higher as compared to the direct time integration of rotation parameters. You [15] explored the corotational frame method on SE(2) and verified that the frame invariance brought by SE(2) is valuable for improving computing efficiency. Rong [16] built a Lie group SE(3) extension of the generalized-alpha time integration method to solve the equations of motion for thin-walled beams in space. Rousso [17] developed a novel feedback linearization technique with Lie group SE(3), and a PID controller involving a coordinate-free pose error function on SE(3) and velocity error on Lie algebra is also built. Flatlandsmo [18] deploys the moving frame method for crane motion with Lie group theory and the work of Elie Cartan.

For the multibody system in space, the system which include three rotations along three axes is most representative. This exploration begins with the Candan rotation, and the mechanism which include three rotation coupling is designed. The kinematics and dynamics model are built by Lie group variational integrator, and the dynamics model is simulated at last. This exploration offers a effective exploration for the geometry method using in the multibody dynamics modeling which include different structures rotation matrix coupling.

2. Cardan rotation and mechanism realization

The Cardan rotation is expressed as in Fig. 1, the inertial frame is $O_0x_0y_0z_0$, $O_0x_1y_1z_1$ is obtained by rotate $O_0x_0y_0z_0$ along the axis O_0z_0 with θ_1 on anticlockwise direction, so O_0z_1 O_0z_0 are coincide together during rotation.

The second step is rotating $O_0x_1y_1z_1$ along O_0y_1 on the anticlockwise direction with θ_2 to obtain $O_0x_2y_2z_2$, O_0y_1 and O_0y_2 are coincide. The third step is rotating $O_0x_2y_2z_2$ along O_0x_2 on the anticlockwise direction with the angle θ_3, O_0x_3 and O_0x_2 are coincide. From the above rotation relation process, the Cardon angle has the following characters when it is used to describe the rotation in space. Firstly, the latter rotation base on the new coordinate which is obtained by the former rotation. Secondly, the rotations must obey the sequence as z, y, x, the orthogonality relation is guaranteed during the rotation. Thirdly, the centers of the rotations are coinciding together. The schematic diagram of mechanism of three axis rotation platform which obey the upper three characters is as Fig. 2. The mechanism has four parts, part *0* is the base frame, part *1* can rotate long the vertical axis of part *0*, part *2* can realize pitching rotation relative part *1*, and part *3* can realize yawn rotation relative to part *2*. The centers of three rotations are coinciding at point *B*. So, the rotation of part *3* can have three degrees of freedom rotation along *B* in space.

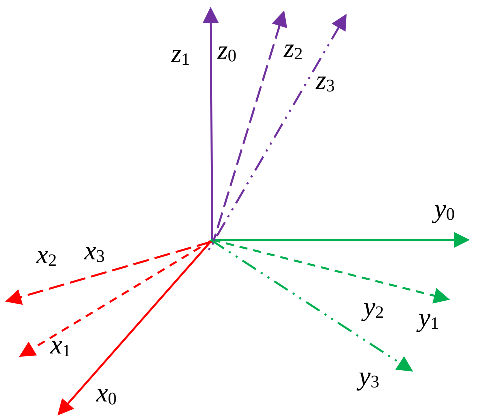

Fig. 1 The rotation relation of Cardon angle

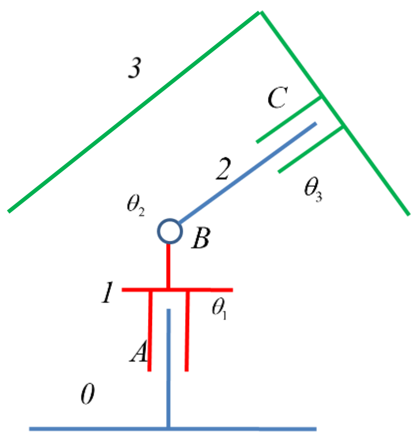

Fig. 2 The mechanism of Cardon rotation realization

The attitude expressions of each parts are as follows. Supposing that the rotation angles of yaw, pitch, roll are $\theta_1, \theta_2, \theta_3$, rotation matrix are $\boldsymbol{R}_z(\theta_1), \boldsymbol{R}_y(\theta_2), \boldsymbol{R}_x(\theta_3)$ respectively. The corner marks of $\boldsymbol{R}$ represent the rotation axis in its own body coordinate. The rotation of each body is analyzed as follows. The horizontal rotating table have one degree of freedom which means $\boldsymbol{R}_1 = \boldsymbol{R}_z$. The pitching table is rotate along the y axis of the horizontal rotating table, so the attitude matrix of pitching table is the combination of two rotating matrix, which means $\boldsymbol{R}_2 = \boldsymbol{R}_z\boldsymbol{R}_y$. Similarly, the attitude of rolling table is the combination of the former two rotation and its own rolling rotation, which means $\boldsymbol{R}_3 = \boldsymbol{R}_z\boldsymbol{R}_y\boldsymbol{R}_x$. From the above analysis, under the driven by three motors, the platform connects with the rolling table realize the 3D rotation in space which can be expressed by the Cardan angles. The concrete expressions of attitude matrixes $\boldsymbol{R}_z\boldsymbol{R}_y\boldsymbol{R}_x$ are as in Eq. (1). In the above derivations, $\boldsymbol{R}_1, \boldsymbol{R}_2, \boldsymbol{R}_3$ are the absolute rotation of rigid bodies, and $\boldsymbol{R}_z, \boldsymbol{R}_y, \boldsymbol{R}_x$ represent the relative rotations.

$$\boldsymbol{R}_z(\theta_1) = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 & 0 \\ \sin\theta_1 & \cos\theta_1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$\boldsymbol{R}_y(\theta_2) = \begin{bmatrix} \cos\theta_2 & 0 & \sin\theta_2 \\ 0 & 1 & 0 \\ -\sin\theta_2 & 0 & \cos\theta_2 \end{bmatrix},$$

$$\boldsymbol{R}_x(\theta_3) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_3 & -\sin\theta_3 \\ 0 & \sin\theta_3 & \cos\theta_3 \end{bmatrix}. \tag{1}$$

3. Kinematics analysis

In actual engineering design, the structure of each body in multibody system always asymmetry, because the design idea and environment condition. So, the mass center position vector doesn't coincide with the rotation axis. According to Fig. 2, a structure design of 3D platform is as in Fig. 3. The pitching driving motor is connecting with the pitching axis by gear, which makes the structure more compact. The inertia parameters of each part of the system can be obtained directly from the model of the platform. The main parts and their parameters of the system are as in Fig. 4. Supposing that the distance between point *A* and point *B* is l_1, as in Fig. 3, so the position vector of point *B* relative to *A* is $\boldsymbol{l}_1 = l_1\boldsymbol{e}_3, \boldsymbol{e}_3 = [0;0;1]$, which means the position is along the z axis. The pose vector of the mass center

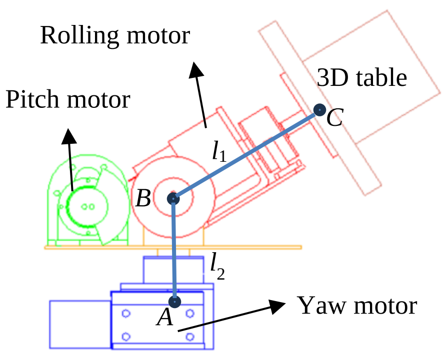

Fig. 3 The 3D rotate platform

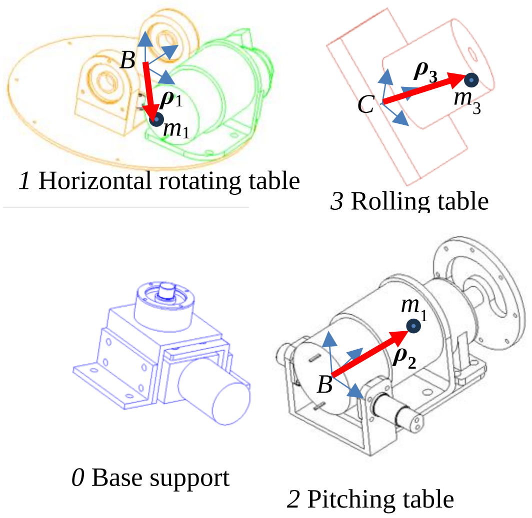

Fig. 4 The main parts of the platform

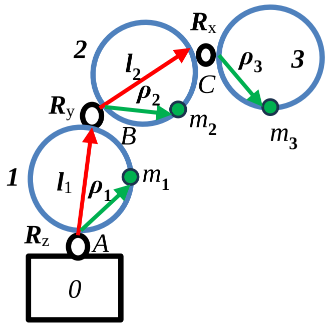

Fig. 5 The topology relation of the system

of table *1* (Horizontal rotating table) in its own body coordinate is ρ_1, the distance from *B* to *C* is l_2, so the position vector of *C* relative to *B* is $\boldsymbol{l}_2 = l_2\boldsymbol{e}_1$ in its own coordinate, the position vector of table *2* (Pitching table) mass center is ρ_2, the position vector of the mass center of table *3* (Rolling table) is ρ_3. The concrete position of mass centres and their position vectors are expressed in Fig. 4. The topology relation of the whole system can be expressed as in Fig. 5.

With the above attitude relation, the kinematic relation of each body in space is derived as follows. The kinematics analysis includes the pose and attitude of mass centers, the velocities and accelerations, which offers the basis for the following dynamics modeling. According to the topology relation in Fig. 5, the position vectors of three tables' mass centers in the inertia coordinate is as in Eq. (2)

$$\begin{aligned}
\boldsymbol{s}_1 &= \boldsymbol{R}_z\boldsymbol{\rho}_1,\\
\boldsymbol{s}_2 &= \boldsymbol{R}_z\left(l_1\boldsymbol{e}_3 + R_y\boldsymbol{\rho}_2\right) = l_1\boldsymbol{R}_z\boldsymbol{e}_3 + \boldsymbol{R}_z\boldsymbol{R}_y\boldsymbol{\rho}_2,\\
\boldsymbol{s}_3 &= \boldsymbol{R}_z\left(l_1\boldsymbol{e}_3 + \boldsymbol{R}_y\left(l_2\boldsymbol{e}_1 + \boldsymbol{R}_x\boldsymbol{\rho}_3\right)\right) =\\
&= l_1\boldsymbol{R}_z\boldsymbol{e}_3 + l_2\boldsymbol{R}_z\boldsymbol{R}_y\boldsymbol{e}_1 + \boldsymbol{R}_z\boldsymbol{R}_y\boldsymbol{R}_x\boldsymbol{\rho}_3. \quad . \qquad (2)
\end{aligned}$$

The velocities relations are derived as follows. The first-order derivatives of relative rotation matrixes $\boldsymbol{R}_z\boldsymbol{R}_y\boldsymbol{R}_x$ satisfies the following characters, as in Eq. (3):

$$\begin{aligned}
\dot{\boldsymbol{R}}_z(\theta_1) &= \omega_z\boldsymbol{R}_z(\theta_1)\boldsymbol{S}(\boldsymbol{e}_3),\\
\dot{\boldsymbol{R}}_y(\theta_2) &= \omega_y\boldsymbol{R}_y(\theta_2)\boldsymbol{S}(\boldsymbol{e}_2),\\
\dot{\boldsymbol{R}}_x(\theta_3) &= \omega_x\boldsymbol{R}_x(\theta_3)\boldsymbol{S}(\boldsymbol{e}_1).
\end{aligned} \qquad (3)$$

In Eq. (3), the concrete expressions of $\boldsymbol{S}(\boldsymbol{e}_i)$ are as follows:

$$\boldsymbol{S}(\boldsymbol{e}_3)=\begin{bmatrix}0 & -1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 0\end{bmatrix},\ \boldsymbol{S}(\boldsymbol{e}_2)=\begin{bmatrix}0 & 0 & 1\\ 0 & 0 & 0\\ -1 & 0 & 0\end{bmatrix},$$

$$\boldsymbol{S}(\boldsymbol{e}_1)=\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & -1\\ 0 & 1 & 0\end{bmatrix}.$$

Take derivation to Eq. (2), the velocities of mass centres are as in Eq. (4)

$$\begin{aligned}
\dot{\boldsymbol{s}}_1 &= \dot{\boldsymbol{R}}_z\boldsymbol{\rho}_1,\\
\dot{\boldsymbol{s}}_2 &= l_1\dot{\boldsymbol{R}}_z\boldsymbol{e}_3 + \dot{\boldsymbol{R}}_z\boldsymbol{R}_y\boldsymbol{\rho}_2 + \boldsymbol{R}_z\dot{\boldsymbol{R}}_y\boldsymbol{\rho}_2,\\
\dot{\boldsymbol{s}}_3 &= l_1\dot{\boldsymbol{R}}_z\boldsymbol{e}_3 + l_2\dot{\boldsymbol{R}}_z\boldsymbol{R}_y\boldsymbol{e}_1 + l_2\boldsymbol{R}_z\dot{\boldsymbol{R}}_y\boldsymbol{e}_1 +\\
&+\dot{\boldsymbol{R}}_z\boldsymbol{R}_y\boldsymbol{R}_x\boldsymbol{\rho}_3 + \boldsymbol{R}_z\dot{\boldsymbol{R}}_y\boldsymbol{R}_x\boldsymbol{\rho}_3 + \boldsymbol{R}_z\boldsymbol{R}_y\dot{\boldsymbol{R}}_x\boldsymbol{\rho}_3\,. \qquad (4)
\end{aligned}$$

Then substituting Eq. (3) into Eq. (4), the mass centre velocities as in Eq. (5):

$$\begin{aligned}
\dot{\boldsymbol{s}}_1 &= \omega_z\boldsymbol{R}_z\boldsymbol{S}_3\boldsymbol{\rho}_1,\\
\dot{\boldsymbol{s}}_2 &= l_1\omega_z\boldsymbol{R}_z\boldsymbol{S}_3\boldsymbol{e}_3 + \omega_z\boldsymbol{R}_z\boldsymbol{S}_3\boldsymbol{R}_y\boldsymbol{\rho}_2 + \omega_y\boldsymbol{R}_z\boldsymbol{R}_y\boldsymbol{S}_2\boldsymbol{\rho}_2,\\
\dot{\boldsymbol{s}}_3 &= l_1\omega_z\boldsymbol{R}_z\boldsymbol{S}_3\boldsymbol{e}_3 + l_2\omega_z\boldsymbol{R}_z\boldsymbol{S}_3\boldsymbol{R}_y\boldsymbol{e}_1 + l_2\boldsymbol{R}_z\dot{\boldsymbol{R}}_y\boldsymbol{e}_1 +\\
&+\omega_z\boldsymbol{R}_z\boldsymbol{S}_3\boldsymbol{R}_y\boldsymbol{R}_x\boldsymbol{\rho}_3 + \omega_y\boldsymbol{R}_z\boldsymbol{R}_y\boldsymbol{S}_2\boldsymbol{R}_x\boldsymbol{\rho}_3 + \omega_x\boldsymbol{R}_z\boldsymbol{R}_y\boldsymbol{R}_x\boldsymbol{S}_1\boldsymbol{\rho}_3. \quad (5)
\end{aligned}$$

Take derivation to Eq. (5), the accelerations of mass centres are as in Eq. (6):

$$\begin{aligned}
\ddot{\boldsymbol{s}}_1 &= \alpha_z\boldsymbol{R}_z\boldsymbol{S}_3\boldsymbol{\rho}_1 + \omega_z^2\boldsymbol{R}_z\boldsymbol{S}_3\boldsymbol{S}_3\boldsymbol{\rho}_1,\\
\ddot{\boldsymbol{s}}_2 &= \alpha_z\boldsymbol{R}_z\boldsymbol{S}_3\left(l_1\boldsymbol{e}_3 + \boldsymbol{R}_y\boldsymbol{\rho}_2\right) + \alpha_y\boldsymbol{R}_z\boldsymbol{R}_y\boldsymbol{S}_2\boldsymbol{\rho}_2 + \omega_y^2\boldsymbol{R}_z\boldsymbol{R}_y\boldsymbol{S}_2\boldsymbol{S}_2\boldsymbol{\rho}_2 + \omega_z^2\boldsymbol{R}_z\boldsymbol{S}_3\boldsymbol{S}_3\left(l_1\boldsymbol{e}_3 + \boldsymbol{R}_y\boldsymbol{\rho}_2\right) + 2\omega_y\omega_z\boldsymbol{R}_z\boldsymbol{S}_3\boldsymbol{R}_y\boldsymbol{S}_2\boldsymbol{\rho}_2,\\
\ddot{\boldsymbol{s}}_3 &= \alpha_x\boldsymbol{R}_z\boldsymbol{R}_y\boldsymbol{R}_x\boldsymbol{S}_1\boldsymbol{\rho}_3 + \alpha_y\boldsymbol{R}_z\boldsymbol{R}_y\boldsymbol{S}_2\left(\boldsymbol{R}_x\boldsymbol{r}_3 + l_2\boldsymbol{e}_1\right) + \alpha_z\boldsymbol{R}_z\boldsymbol{S}_3\left(l_1\boldsymbol{e}_3 + \boldsymbol{R}_y\boldsymbol{R}_x\boldsymbol{r}_3 + l_2\boldsymbol{R}_y\boldsymbol{e}_1\right) + \omega_z^2\boldsymbol{R}_z\boldsymbol{S}_3\boldsymbol{S}_3\left(l_2\boldsymbol{R}_y\boldsymbol{e}_1 + l_1\boldsymbol{e}_3 + \boldsymbol{R}_y\boldsymbol{R}_x\boldsymbol{\rho}_3\right) +\\
&+\omega_y^2\boldsymbol{R}_z\boldsymbol{R}_y\boldsymbol{S}_2\boldsymbol{S}_2\left(l_2\boldsymbol{e}_1 + \boldsymbol{R}_x\boldsymbol{\rho}_3\right) + \omega_x^2\boldsymbol{R}_z\boldsymbol{R}_y\boldsymbol{R}_x\boldsymbol{S}_1\boldsymbol{S}_1\boldsymbol{\rho}_3 + 2l_2\omega_y\omega_z\boldsymbol{R}_z\boldsymbol{S}_3\boldsymbol{R}_y\boldsymbol{S}_2\boldsymbol{e}_1 + 2\omega_z\omega_y\boldsymbol{R}_z\boldsymbol{S}_3\boldsymbol{R}_y\boldsymbol{S}_2\boldsymbol{R}_x\boldsymbol{\rho}_3 +\\
&+2\omega_z\omega_x\boldsymbol{R}_z\boldsymbol{S}_3\boldsymbol{R}_y\boldsymbol{R}_x\boldsymbol{S}_1\boldsymbol{\rho}_3 + 2\omega_y\omega_x\boldsymbol{R}_z\boldsymbol{R}_y\boldsymbol{S}_2\boldsymbol{R}_x\boldsymbol{S}_1\boldsymbol{\rho}_3. \qquad (6)
\end{aligned}$$

From the above analysis, the kinematics relation with the relative rotation matrix has a complex structure, which bring difficulties in following dynamics modelling, so some new expression methods should be considered. If the motion of the bodies is expressed by absolute attitudes in space, the absolute mass centre positions are expressed as in Eq. (7):

$$\boldsymbol{s}_1 = \boldsymbol{R}_1\boldsymbol{\rho}_1,$$
$$\boldsymbol{s}_2 = l_1\boldsymbol{R}_1\boldsymbol{e}_3 + \boldsymbol{R}_2\boldsymbol{\rho}_2,$$
$$\boldsymbol{s}_3 = l_1\boldsymbol{R}_1\boldsymbol{e}_3 + l_2\boldsymbol{R}_2\boldsymbol{e}_1 + \boldsymbol{R}_3\boldsymbol{\rho}_3. \quad (7)$$

Supposing that the absolute angular velocities of body 1, 2, 3 are $\boldsymbol{\omega}_1$, $\boldsymbol{\omega}_2$, $\boldsymbol{\omega}_3$ respectively. The velocities of mass centres are as in Eq. (8):

$$\dot{\boldsymbol{s}}_1 = \boldsymbol{R}_1\boldsymbol{S}(\boldsymbol{\omega}_1)\boldsymbol{\rho}_1,$$
$$\dot{\boldsymbol{s}}_2 = l_1\boldsymbol{R}_1\boldsymbol{S}(\boldsymbol{\omega}_1)\boldsymbol{e}_3 + \boldsymbol{R}_2\boldsymbol{S}(\boldsymbol{\omega}_2)\boldsymbol{\rho}_2,$$
$$\dot{\boldsymbol{s}}_3 = l_1\boldsymbol{R}_1\boldsymbol{S}(\boldsymbol{\omega}_1)\boldsymbol{e}_3 + l_2\boldsymbol{R}_2\boldsymbol{S}(\boldsymbol{\omega}_2)\boldsymbol{e}_1 + \boldsymbol{R}_3\boldsymbol{S}(\boldsymbol{\omega}_3)\boldsymbol{\rho}_3. \quad (8)$$

Similarly, the accelerations which expressed by absolute rotation matrix are as in Eq. (9):

$$\ddot{\boldsymbol{s}}_1 = \boldsymbol{R}_1\boldsymbol{S}^2(\boldsymbol{\omega}_1)\boldsymbol{\rho}_1 + \boldsymbol{R}_1\boldsymbol{S}(\boldsymbol{\alpha}_1)\boldsymbol{\rho}_1,$$
$$\ddot{\boldsymbol{s}}_2 = l_1\boldsymbol{R}_1\boldsymbol{S}^2(\boldsymbol{\omega}_1)\boldsymbol{e}_3 + l_1\boldsymbol{R}_1\boldsymbol{S}(\boldsymbol{\alpha}_1)\boldsymbol{e}_3 +$$
$$+\boldsymbol{R}_2\boldsymbol{S}^2(\boldsymbol{\omega}_2)\boldsymbol{\rho}_2 + \boldsymbol{R}_2\boldsymbol{S}(\boldsymbol{\alpha}_2)\boldsymbol{\rho}_2,$$
$$\ddot{\boldsymbol{s}}_3 = l_1\boldsymbol{R}_1\boldsymbol{S}^2(\boldsymbol{\omega}_1)\boldsymbol{e}_3 + l_1\boldsymbol{R}_1\boldsymbol{S}(\boldsymbol{\alpha}_1)\boldsymbol{e}_3 +$$
$$+l_2\boldsymbol{R}_2\boldsymbol{S}^2(\boldsymbol{\omega}_2)\boldsymbol{e}_1 + l_2\boldsymbol{R}_2\boldsymbol{S}(\boldsymbol{\alpha}_2)\boldsymbol{e}_1 +$$
$$+\boldsymbol{R}_3\boldsymbol{S}^2(\boldsymbol{\omega}_3)\boldsymbol{\rho}_3 + \boldsymbol{R}_3\boldsymbol{S}(\boldsymbol{\alpha}_3)\boldsymbol{\rho}_3. \quad (9)$$

The relations of absolute and relative angular velocities are derived in following. According to $\boldsymbol{R}_1 = \boldsymbol{R}_z$, so the absolute angular velocity of body *1* in space is $\boldsymbol{\omega}_1 = \dot{\theta}_z\boldsymbol{e}_3 = \omega_z\boldsymbol{e}_3$. According to $\boldsymbol{R}_2 = \boldsymbol{R}_z\boldsymbol{R}_y$, the derivation of it is as in Eq. (10):

$$\dot{\boldsymbol{R}}_2 = \dot{\boldsymbol{R}}_z\boldsymbol{R}_y + \boldsymbol{R}_z\dot{\boldsymbol{R}}_y =$$
$$= \boldsymbol{R}_z\boldsymbol{S}(\boldsymbol{\omega}_z)\boldsymbol{R}_y + \boldsymbol{R}_z\boldsymbol{R}_y\boldsymbol{S}(\boldsymbol{\omega}_y) =$$
$$= \boldsymbol{R}_z\boldsymbol{R}_y\boldsymbol{R}_y^T\boldsymbol{S}(\boldsymbol{\omega}_z)\boldsymbol{R}_y + \boldsymbol{R}_z\boldsymbol{R}_y\boldsymbol{S}(\boldsymbol{\omega}_y). \quad (10)$$

According to $S(\boldsymbol{R}^T\boldsymbol{x}) = \boldsymbol{R}^T S(\boldsymbol{x})\boldsymbol{R}$, Eq. (10) can be simplified as in Eq. (11):

$$\dot{\boldsymbol{R}}_2 = \boldsymbol{R}_z\boldsymbol{R}_y\boldsymbol{S}(\boldsymbol{R}_y^T\boldsymbol{e}_3\omega_z) + \boldsymbol{R}_z\boldsymbol{R}_y\boldsymbol{S}(\boldsymbol{e}_2\omega_y) =$$
$$= \boldsymbol{R}_z\boldsymbol{R}_y\boldsymbol{S}(\boldsymbol{R}_y^T\boldsymbol{e}_3\omega_z + \boldsymbol{e}_2\omega_y). \quad (11)$$

So $\boldsymbol{\omega}_2 = \omega_z\boldsymbol{R}_y^T\boldsymbol{e}_3 + \omega_y\boldsymbol{e}_2$. Similarly, the absolute rotation matrix of the platform is $\boldsymbol{R}_3 = \boldsymbol{R}_z\boldsymbol{R}_y\boldsymbol{R}_x$. Take derivation to it and the result is as in Eq. (12):

$$\dot{\boldsymbol{R}}_3 = \dot{\boldsymbol{R}}_z\boldsymbol{R}_y\boldsymbol{R}_x + \boldsymbol{R}_z\dot{\boldsymbol{R}}_y\boldsymbol{R}_x + \boldsymbol{R}_z\boldsymbol{R}_y\dot{\boldsymbol{R}}_x =$$
$$= \boldsymbol{R}_z\boldsymbol{R}_y\boldsymbol{R}_x\boldsymbol{S}\left(\omega_z(\boldsymbol{R}_y\boldsymbol{R}_x)^T\boldsymbol{e}_3 + \omega_y\boldsymbol{R}_x{}^T\boldsymbol{e}_2 + \omega_x\boldsymbol{e}_1\right). \quad (12)$$

So $\boldsymbol{\omega}_3 = \omega_z(\boldsymbol{R}_y\boldsymbol{R}_x)^T\boldsymbol{e}_3 + \omega_y\boldsymbol{R}_x{}^T\boldsymbol{e}_2 + \omega_x\boldsymbol{e}_1$ .According to the expressions of $\boldsymbol{\omega}_1, \boldsymbol{\omega}_2, \boldsymbol{\omega}_3$, the angular accelerations of each body are as in Eq. (13):

$$\boldsymbol{\alpha}_1 = \alpha_z\boldsymbol{e}_3,$$
$$\boldsymbol{\alpha}_2 = \alpha_z\boldsymbol{R}_y^T\boldsymbol{e}_3 - \omega_y\omega_z\boldsymbol{S}_y\boldsymbol{R}_y^T\boldsymbol{e}_3 + \alpha_y\boldsymbol{e}_2,$$
$$\boldsymbol{\alpha}_3 = \alpha_z(\boldsymbol{R}_y\boldsymbol{R}_x)^T\boldsymbol{e}_3 + \omega_y\omega_z(\boldsymbol{R}_y\boldsymbol{S}_y\boldsymbol{R}_x)^T\boldsymbol{e}_3 +$$
$$+\omega_x\omega_z(\boldsymbol{R}_y\boldsymbol{R}_x\boldsymbol{S}_x)^T\boldsymbol{e}_3 + \alpha_y\boldsymbol{R}_x{}^T\boldsymbol{e}_2 -$$
$$-\omega_y\omega_x\boldsymbol{S}_x\boldsymbol{R}_x{}^T\boldsymbol{e}_2 + \alpha_x\boldsymbol{e}_1. \quad (13)$$

4. Dynamics Modelling

There are three rigid bodies in the above platform system. Each motion is the combination of the mass centre movement and the rotation along the mass centre. For convenient, the base points of each body are coinciding with the hinge joint of the body connects with the upper stage body. In this platform, all of the bodies are rotating along the point B in space. The motion of each body is divided into two parts, the movement of the mass centre and the rotation along the mass centre. The Lagrange function need to be built firstly. Supposing that the masses of three bodies are m_1, m_2, m_3 respectively, and the moment of inertia are $\boldsymbol{J}_1$, $\boldsymbol{J}_2$, $\boldsymbol{J}_3$. The Lagrange function of the system is as Eq. (14):

$$L_1 = \frac{1}{2}\boldsymbol{\omega}^T\boldsymbol{J}\boldsymbol{\omega} + \frac{1}{2}\dot{\boldsymbol{s}}^T\boldsymbol{M}\dot{\boldsymbol{s}} - g\boldsymbol{E}^T\boldsymbol{M}\boldsymbol{s}. \quad (14)$$

The elements in Eq. (14) are written as matrix type as follows, which makes Lagrange function simple:

$$\boldsymbol{J} = \begin{bmatrix} \boldsymbol{J}_1 & \boldsymbol{0}_{3\times3} & \boldsymbol{0}_{3\times3} \\ \boldsymbol{0}_{3\times3} & \boldsymbol{J}_2 & \boldsymbol{0}_{3\times3} \\ \boldsymbol{0}_{3\times3} & \boldsymbol{0}_{3\times3} & \boldsymbol{J}_3 \end{bmatrix}, \boldsymbol{\omega} = \begin{bmatrix} \boldsymbol{\omega}_1 \\ \boldsymbol{\omega}_2 \\ \boldsymbol{\omega}_3 \end{bmatrix},$$
$$\boldsymbol{M} = \begin{bmatrix} m_1\boldsymbol{I}_{3\times3} & \boldsymbol{0}_{3\times3} & \boldsymbol{0}_{3\times3} \\ \boldsymbol{0}_{3\times3} & m_2\boldsymbol{I}_{3\times3} & \boldsymbol{0}_{3\times3} \\ \boldsymbol{0}_{3\times3} & \boldsymbol{0}_{3\times3} & m_3\boldsymbol{I}_{3\times3} \end{bmatrix}, \boldsymbol{E} = \begin{bmatrix} \boldsymbol{e}_3 \\ \boldsymbol{e}_3 \\ \boldsymbol{e}_3 \end{bmatrix}^T \boldsymbol{s} = \begin{bmatrix} \boldsymbol{s}_1 \\ \boldsymbol{s}_2 \\ \boldsymbol{s}_3 \end{bmatrix}.$$

The dynamics model of the system is derived as follows. Taking variation to the angular velocity and the results is as in Eq. (15):

$$\delta_\omega L = \boldsymbol{D}_\omega L(\boldsymbol{R},\omega)\cdot\delta\omega. \quad (15)$$

According to the kinematics relation in Eq. (8), Eq. (15) can be written as Eq. (16):

$$\boldsymbol{D}_\omega L(\boldsymbol{R},\omega) = (\boldsymbol{J}\boldsymbol{\omega})^T\delta\boldsymbol{\omega} + (\boldsymbol{M}\dot{\boldsymbol{s}})^T\delta\dot{\boldsymbol{s}}. \quad (16)$$

According to $\boldsymbol{\omega}_1 = \omega_z\boldsymbol{e}_3$, $\boldsymbol{\omega}_2 = \omega_z\boldsymbol{R}_y^T\boldsymbol{e}_3 + \omega_y\boldsymbol{e}_2$, $\boldsymbol{\omega}_3 = \omega_z(\boldsymbol{R}_y\boldsymbol{R}_x)^T\boldsymbol{e}_3 + \omega_y\boldsymbol{R}_x{}^T\boldsymbol{e}_2 + \omega_x\boldsymbol{e}_1$, the variation relation of absolute and relative angular velocity is as Eq. (17):

$$\delta\boldsymbol{\omega} = \boldsymbol{Q}_1\delta\boldsymbol{\omega}_C. \quad (17)$$

In Eq. (17), $\boldsymbol{\omega} = [\boldsymbol{\omega}_1;\boldsymbol{\omega}_2;\boldsymbol{\omega}_3]$ is the absolute angular velocity vector, and $\boldsymbol{\omega}_C$ is the relative angular velocity vector. The concrete expressions of middle parameters in Eq. (17) are as follows:

$$Q_1 = \begin{bmatrix} e_3 & 0_{3\times1} & 0_{3\times1} \\ R_y^T e_3 & e_2 & 0_{3\times1} \\ \left(R_y R_x\right)^T e_3 & R_x{}^T e_2 & e_1 \end{bmatrix}, \ \delta\omega_C = \begin{bmatrix} \delta\omega_z \\ \delta\omega_y \\ \delta\omega_x \end{bmatrix}.$$

Take variation to $\dot{s}$, and the result is as Eq. (18):

$$\delta\dot{s} = \begin{bmatrix} \delta\dot{s}_1 \\ \delta\dot{s}_2 \\ \delta\dot{s}_3 \end{bmatrix} = -Q_2\delta\omega . \tag{18}$$

In Eq. (18), the middle parameter Q_2 is as follows:

$$Q_2 = \begin{bmatrix} R_1 S(\rho_1) & 0_{3\times3} & 0_{3\times3} \\ l_1 R_1 S(e_3) & R_2 S(\rho_2) & 0_{3\times3} \\ l_1 R_1 S(e_3) & l_2 R_2 S(e_1) & R_3 S(\rho_3) \end{bmatrix}.$$

Substituting Eq. (17) into Eq. (18), and the result is as Eq. (19):

$$\delta\dot{s} = -Q_2 Q_1 \delta\omega_C . \tag{19}$$

According to Hamilton theory the variation of angular velocity to Lagrange function equal to the angular momentum of the system, as in Eq. (20):

$$\Pi = D_\omega L(R,\omega). \tag{20}$$

Substituting Eq. (17) and Eq. (19) into Eq. (16), and the relative momentum which expressed by the relative angular velocity is as in Eq. (21):

$$\begin{aligned} \Pi_C &= \left((J\omega)^T Q_1 - (M\dot{s})^T Q_2 Q_1 \right)^T = \\ &= Q_1^T \left(J\omega - Q_2{}^T M\dot{s} \right). \end{aligned} \tag{21}$$

The absolute momentum of the system which expressed by the absolute angular velocities is as Eq. (22):

$$\Pi = \left((J\omega)^T - (M\dot{s})^T Q_2 \right)^T = J\omega - Q_2{}^T M\dot{s} . \tag{22}$$

In space rotation, the inertia moment also includes the coupling part which expressed by the angular velocity and angular momentum, as Eq. (23):

$$ad_\omega^* \cdot D_\omega L(R,\omega) = S(\omega) \cdot D_\omega L(R,\omega) . \tag{23}$$

If the joint rotation has a single freedom, the coupling part satisfy the following character as in Eq. (24):

$$ad_\omega^* \cdot D_\omega L(R,\omega) = 0 . \tag{24}$$

So, in the 3D table, if the rotation of the rigid bodies is expressed by relative angular velocity ω_C, Eq. (24) is satisfied, if the absolute one as ω, Eq. (23) is satisfied. Substitute Eq. (22) in to Eq. (23), the coupling part under the absolute one is as Eq. (25):

$$ad_\omega^* \cdot D_\omega L(R,\omega) = S(\omega) \cdot \left(J\omega - Q_2{}^T M\dot{s} \right). \tag{25}$$

In Eq. (25), $S(\omega)$ is the combination of skew matrixes which is expressed as follows:

$$S(\omega) = \begin{bmatrix} S(\omega_1) & 0_{3\times3} & 0_{3\times3} \\ 0_{3\times3} & S(\omega_2) & 0_{3\times3} \\ 0_{3\times3} & 0_{3\times3} & S(\omega_3) \end{bmatrix}.$$

At last, the inertia moment which lead by the potential energy is as follows. Take variation to R in Lagrange function L, the process is expressed as Eq. (26):

$$\delta_R L = D_R L(R,\omega) \cdot \eta . \tag{26}$$

In Eq. (19), is a row vector, take transposition to it, as in Eq. (27):

$$T_e^* L_R \cdot \mathbf{D}_R L(R,\omega) = \left(\mathbf{D}_R L(R,\omega) \right)^T . \tag{27}$$

According to Eq. (14), the variation to L, the result is as Eq. (28):

$$\begin{aligned} &D_R L(R,\omega) \cdot \eta = (J\omega)^T \delta_R \omega + \\ &+ (M\dot{s})^T \delta_R \dot{s} - g E^T M \delta_R s. \end{aligned} \tag{28}$$

Solve $\delta_R \omega, \delta_R \dot{s}, \delta_R s$ respectively. Take variation to rotation matrix and angular velocity, and the results are as in Eq. (29) and Eq. (30):

$$\begin{aligned} &\delta R_1 = R_z S(e_3)\eta_z, \\ &\delta R_2 = \eta_z R_z S(e_3) R_y + \eta_y R_z R_y S(e_2), \\ &\delta R_3 = \eta_z R_z S(e_3) R_y R_x + \eta_y R_z R_y S(e_2) R_x + \\ &\quad + \eta_x R_z R_y R_x S(e_1), \end{aligned} \tag{29}$$

$$\delta_R \omega = Q_3 \eta_C . \tag{30}$$

The expressions of middle parameters of Q_3 and η_C are as follows:

$$Q_3 = \begin{bmatrix} 0_{3\times1} & 0_{3\times1} & 0_{3\times1} \\ 0_{3\times1} & -\omega_z S(e_2) R_y^T e_3 & 0_{3\times1} \\ 0_{3\times1} & \omega_z \left(R_y S(e_2) R_x \right)^T e_3 & \begin{matrix} \omega_z \left(R_y R_x S(e_1) \right)^T e_3 \\ -\omega_y S(e_1) R_x{}^T e_2 \end{matrix} \end{bmatrix},$$

$$\eta_C = \left[\eta_z ; \ \eta_y ; \ \eta_x \right].$$

Take variation to the velocity of mass centre, as in Eq. (31):

$$\delta_R \dot{s} = -Q_4 \eta_C - Q_2 \delta_R \omega = -\left(Q_4 + Q_2 Q_3 \right) \eta_C . \tag{31}$$

The expressions of middle parameters of Q_4 is as follows:

$$\boldsymbol{Q}_4=\begin{bmatrix} Q_4^{11} & \boldsymbol{0}_{3\times1} & \boldsymbol{0}_{3\times1} \\ Q_4^{21} & Q_4^{22} & \boldsymbol{0}_{3\times1} \\ Q_4^{31} & Q_4^{32} & Q_4^{33} \end{bmatrix},$$

$$Q_4^{11}=\boldsymbol{R}_1\boldsymbol{S}(\boldsymbol{e}_3)\boldsymbol{S}(\boldsymbol{\rho}_1)\boldsymbol{\omega}_1,$$

$$Q_4^{21}=\boldsymbol{R}_1\boldsymbol{S}(\boldsymbol{e}_3)\big(l_1\boldsymbol{S}(\boldsymbol{e}_3)\boldsymbol{\omega}_1+\boldsymbol{R}_y\boldsymbol{S}(\boldsymbol{\rho}_2)\boldsymbol{\omega}_2\big),$$

$$Q_4^{31}=\boldsymbol{R}_1\boldsymbol{S}(\boldsymbol{e}_3)$$
$$\big(l_1\boldsymbol{S}(\boldsymbol{e}_3)\boldsymbol{\omega}_1+l_2\boldsymbol{R}_y\boldsymbol{S}(\boldsymbol{e}_1)\boldsymbol{\omega}_2+\boldsymbol{R}_y\boldsymbol{R}_x\boldsymbol{S}(\boldsymbol{\rho}_3)\boldsymbol{\omega}_3\big),$$

$$Q_4^{22}=\boldsymbol{R}_2\boldsymbol{S}(\boldsymbol{e}_2)\boldsymbol{S}(\boldsymbol{\rho}_2)\boldsymbol{\omega}_2,$$

$$Q_4^{32}=\boldsymbol{R}_2\boldsymbol{S}(\boldsymbol{e}_2)\big(l_2\boldsymbol{S}(\boldsymbol{e}_1)\boldsymbol{\omega}_2+\boldsymbol{R}_x\boldsymbol{S}(\boldsymbol{\rho}_3)\boldsymbol{\omega}_3\big),$$

$$Q_4^{33}=\boldsymbol{R}_3\boldsymbol{S}(\boldsymbol{e}_1)\boldsymbol{S}(\boldsymbol{\rho}_3)\boldsymbol{\omega}_3.$$

Take variation to the displacement vector of mass centre, and the result is as Eq. (32):

$$\delta_R\boldsymbol{s}=\boldsymbol{Q}_5\boldsymbol{\eta}_C\,. \tag{32}$$

The expressions of middle parameters of $\boldsymbol{Q}_5$ is as follows:

$$\boldsymbol{Q}_5=\begin{bmatrix} \boldsymbol{Q}_{511} & \boldsymbol{0}_{3\times1} & \boldsymbol{0}_{3\times1} \\ \boldsymbol{Q}_{521} & \boldsymbol{Q}_{522} & \boldsymbol{0}_{3\times1} \\ \boldsymbol{Q}_{531} & \boldsymbol{Q}_{532} & \boldsymbol{Q}_{533} \end{bmatrix},$$

$$\boldsymbol{Q}_{511}=\boldsymbol{R}_z\boldsymbol{S}(\boldsymbol{e}_3)\boldsymbol{\rho}_1,$$

$$\boldsymbol{Q}_{521}=\boldsymbol{R}_z\boldsymbol{S}(\boldsymbol{e}_3)\big(l_1\boldsymbol{e}_3+\boldsymbol{R}_y\boldsymbol{\rho}_2\big),$$

$$\boldsymbol{Q}_{531}=\boldsymbol{R}_z\boldsymbol{S}(\boldsymbol{e}_3)\big(l_1\boldsymbol{e}_3+l_2\boldsymbol{R}_y\boldsymbol{e}_1+\boldsymbol{R}_y\boldsymbol{R}_x\boldsymbol{\rho}_3\big),$$

$$\boldsymbol{Q}_{522}=\boldsymbol{R}_2\boldsymbol{S}(\boldsymbol{e}_2)\boldsymbol{\rho}_2,$$

$$\boldsymbol{Q}_{532}=\boldsymbol{R}_2\boldsymbol{S}(\boldsymbol{e}_2)\big(l_2\boldsymbol{e}_1+\boldsymbol{R}_x\boldsymbol{\rho}_3\big),$$

$$\boldsymbol{Q}_{533}=\boldsymbol{R}_3\boldsymbol{S}(\boldsymbol{e}_1)\boldsymbol{\rho}_3.$$

According to Eq. (30), Eq. (31) and Eq. (32), the inertia moment lead by gravity is as in Eq. (33):

$$\boldsymbol{T}_e^*\boldsymbol{L}_{\boldsymbol{R}}\cdot\boldsymbol{D}_{\boldsymbol{R}}L(\boldsymbol{R},\omega)=$$
$$=\boldsymbol{Q}_3^{\,T}\boldsymbol{J}\boldsymbol{\omega}-(\boldsymbol{Q}_4+\boldsymbol{Q}_2\boldsymbol{Q}_3)^T\boldsymbol{M}\dot{\boldsymbol{s}}-g\boldsymbol{Q}_5^{\,T}\boldsymbol{M}\boldsymbol{E}. \tag{33}$$

When the system is expressed by absolute rotation, the inertia moment can be written as in Eq. (34):

$$\boldsymbol{T}_e^*\boldsymbol{L}_{\boldsymbol{R}}\cdot\boldsymbol{D}_{\boldsymbol{R}}L(\boldsymbol{R},\omega)=(\boldsymbol{M}\dot{\boldsymbol{s}})^T\,\delta_R\dot{\boldsymbol{s}}-g\boldsymbol{E}^T\boldsymbol{M}\delta_R\boldsymbol{s}\,. \tag{34}$$

Take variation to the position vector of mass centre, and the result is as Eq. (35):

$$\delta_R\boldsymbol{s}=\begin{bmatrix}\delta_R\boldsymbol{s}_1\\ \delta_R\boldsymbol{s}_2\\ \delta_R\boldsymbol{s}_3\end{bmatrix}=-\boldsymbol{Q}_2\boldsymbol{\eta}=-\boldsymbol{Q}_2\begin{bmatrix}\boldsymbol{\eta}_1\\ \boldsymbol{\eta}_2\\ \boldsymbol{\eta}_3\end{bmatrix}. \tag{35}$$

Take variation to velocity vector of mass centre, and the result is as Eq. (36):

$$\delta_R\dot{\boldsymbol{s}}=\begin{bmatrix}\delta_R\dot{\boldsymbol{s}}_1\\ \delta_R\dot{\boldsymbol{s}}_2\\ \delta_R\dot{\boldsymbol{s}}_3\end{bmatrix}=-\boldsymbol{Q}_6\boldsymbol{\eta}\,. \tag{36}$$

The middle parameter $\boldsymbol{Q}_6$ is expressed as follows:

$$\boldsymbol{Q}_6=\begin{bmatrix} \boldsymbol{R}_1\boldsymbol{S}\big(\boldsymbol{S}(\boldsymbol{\omega}_1)\boldsymbol{\rho}_1\big) & \boldsymbol{0}_{3\times3} & \boldsymbol{0}_{3\times3} \\ l_1\boldsymbol{R}_1\boldsymbol{S}\big(\boldsymbol{S}(\boldsymbol{\omega}_1)\boldsymbol{e}_3\big) & \boldsymbol{R}_2\boldsymbol{S}\big(\boldsymbol{S}(\boldsymbol{\omega}_2)\boldsymbol{\rho}_2\big) & \boldsymbol{0}_{3\times3} \\ l_1\boldsymbol{R}_1\boldsymbol{S}\big(\boldsymbol{S}(\boldsymbol{\omega}_1)\boldsymbol{e}_3\big) & l_2\boldsymbol{R}_2\boldsymbol{S}\big(\boldsymbol{S}(\boldsymbol{\omega}_2)\boldsymbol{e}_1\big) & \boldsymbol{R}_3\boldsymbol{S}\big(\boldsymbol{S}(\boldsymbol{\omega}_3)\boldsymbol{\rho}_3\big) \end{bmatrix}.$$

According to Eq. (35) and Eq. (36), the inertia moment under the absolute rotation is as Eq. (37):

$$\boldsymbol{T}_e^*\boldsymbol{L}_{\boldsymbol{R}}\cdot\boldsymbol{D}_{\boldsymbol{R}}L(\boldsymbol{R},\omega)=g\boldsymbol{Q}_2^{\,T}\boldsymbol{M}^T\boldsymbol{E}-\boldsymbol{Q}_6^{\,T}\boldsymbol{M}\dot{\boldsymbol{s}}\,. \tag{37}$$

The dynamics equation of the system is as Eq. (38):

$$\frac{d}{dt}\boldsymbol{D}_\omega L(\boldsymbol{R},\omega)-ad_\omega^*\cdot\boldsymbol{D}_\omega L(\boldsymbol{R},\omega)-$$
$$-\boldsymbol{T}_e^*\boldsymbol{L}_{\boldsymbol{R}}\cdot\boldsymbol{D}_{\boldsymbol{R}}L(\boldsymbol{R},\omega)=\boldsymbol{0}. \tag{38}$$

If the system is working on planar, the dynamics equation can be simplified as Eq. (39):

$$\frac{d}{dt}\boldsymbol{D}_\omega L(\boldsymbol{R},\omega)-\boldsymbol{T}_e^*\boldsymbol{L}_{\boldsymbol{R}}\cdot\boldsymbol{D}_{\boldsymbol{R}}L(\boldsymbol{R},\omega)=\boldsymbol{0}\,. \tag{39}$$

From Eq. (38) and Eq. (39), if the dynamics equation of the system is expressed as the Lagrange type, the difference derivation is needed to the concrete expression of momentum, which will lead to the complexity of the system. So according to the Legendre transformation, the momentum can be used as a variable of the system, which can reduce the complexity of the system obviously. So according to Eq. (16), Eq. (22) can be transformed to be a dynamics equation with the Hamilton type, as in Eq. (40):

$$\dot{\boldsymbol{\Pi}}-ad_\omega^*\boldsymbol{\Pi}-\boldsymbol{T}_e^*\boldsymbol{L}_{\boldsymbol{g}}\cdot\boldsymbol{D}_{\boldsymbol{R}}L(\boldsymbol{R},\omega)=0\,. \tag{40}$$

According to Eq. (40), the dynamics equations which express by absolute and relative parameters are as Eq. (41) and Eq. (42) respectively:

$$\dot{\boldsymbol{\Pi}}+\big(\boldsymbol{S}(\boldsymbol{\Pi})-\boldsymbol{Q}_6^{\,T}\boldsymbol{M}\boldsymbol{Q}_2\big)\boldsymbol{\omega}-g\boldsymbol{Q}_2^{\,T}\boldsymbol{M}^T\boldsymbol{E}=0,$$
$$\boldsymbol{\Pi}=\big(\boldsymbol{J}+\boldsymbol{Q}_2^{\,T}\boldsymbol{M}\boldsymbol{Q}_2\big)\boldsymbol{\omega}, \tag{41}$$

$$\dot{\boldsymbol{\Pi}}_C-\Big(\boldsymbol{Q}_3^{\,T}\boldsymbol{J}\boldsymbol{Q}_1+(\boldsymbol{Q}_4+\boldsymbol{Q}_2\boldsymbol{Q}_3)^T\boldsymbol{M}\boldsymbol{Q}_2\boldsymbol{Q}_1\Big)\boldsymbol{\omega}_C+$$
$$+g\boldsymbol{Q}_5^{\,T}\boldsymbol{M}\boldsymbol{E}=0,$$
$$\boldsymbol{\Pi}_C=\boldsymbol{Q}_1^T\big(\boldsymbol{J}+\boldsymbol{Q}_2^{\,T}\boldsymbol{M}\boldsymbol{Q}_2\big)\boldsymbol{Q}_1\boldsymbol{\omega}_C. \tag{42}$$

According to Eq. (41) and Eq. (42), the Lagrange dynamics equation which expressed by the angular velocity is transformed to be the Hamilton dynamics equation, and the momentum is used as the variables. For the difference calculation is not needed in the Hamilton equation, the complexity of the system is reduced, which is convenient to the realization of programming.

For the multibody system in planar, the rotation coupling relation can be expressed as absolute and relative types which has same complexity. However, for the multi-body system in space, this equal relation is not suit. Comparing Eq. (41) and Eq. (42), the dynamics equation with absolute parameter is more simple than relative one, but the motion relation is destroyed in the absolute one. As in Eq. (41), the rotation matrix and angular velocities of each body has the same structure, which is different with the actual motion. The absolute and relative type dynamics equations of multibody system with different rotation structures in space are essentially different, which is the most obvious characters distinguish with the system in planar. As summary, when the rotation axis of joint in multi-body system has different rotation directions, the rotation matrix has different structures, so the absolute expression is not suit to use in the dynamics modelling.

5. Numerical computation

The computation of dynamics equation of the multi-body system is based on the combination of dynamics equation and the kinematics equation. The equation is an ordinary differential equation, and the number of the variables equal to the number of dimensions. Under the frame of Lie group and Lie algebra expressions, the kinematics equation has two expression types, the first is Lie group one which using the attitude matrix as parameter, the second is the Lie algebra one, which use the rotation angle as the parameter. The angular velocity $\boldsymbol{\omega}_C$ in Eq. (42) can be replaced by the expression of $\boldsymbol{\Pi}_C$ .The kinematic equation of the system can be written as Eq. (44), and the concrete expressions of $\boldsymbol{R}$ and $\boldsymbol{S}(\boldsymbol{\omega})$ are as follows:

$$\dot{\boldsymbol{R}}_C = \boldsymbol{R}_C \boldsymbol{S}(\boldsymbol{\omega}_C) . \tag{43}$$

Here:

$$\boldsymbol{R}_C = \begin{bmatrix} \boldsymbol{R}_z & \boldsymbol{0}_{3\times3} & \boldsymbol{0}_{3\times3} \\ \boldsymbol{0}_{3\times3} & \boldsymbol{R}_y & \boldsymbol{0}_{3\times3} \\ \boldsymbol{0}_{3\times3} & \boldsymbol{0}_{3\times3} & \boldsymbol{R}_x \end{bmatrix},$$

$$\boldsymbol{S}(\boldsymbol{\omega}_C) = \begin{bmatrix} \boldsymbol{S}(\boldsymbol{\omega}_z) & \boldsymbol{0}_{3\times3} & \boldsymbol{0}_{3\times3} \\ \boldsymbol{0}_{3\times3} & \boldsymbol{S}(\boldsymbol{\omega}_y) & \boldsymbol{0}_{3\times3} \\ \boldsymbol{0}_{3\times3} & \boldsymbol{0}_{3\times3} & \boldsymbol{S}(\boldsymbol{\omega}_x) \end{bmatrix}.$$

The angular velocity in Eq. (44) can also be replaced by $\boldsymbol{\Pi}_C$:

$$\begin{aligned} \dot{\boldsymbol{\Pi}}_C &= \left(\boldsymbol{Q}_3^T \boldsymbol{J}\boldsymbol{Q}_1 + (\boldsymbol{Q}_4 + \boldsymbol{Q}_2\boldsymbol{Q}_3)^T \boldsymbol{M}\boldsymbol{Q}_2\boldsymbol{Q}_1\right) \\ &\quad \left(\boldsymbol{Q}_1^T \left(\boldsymbol{J} + \boldsymbol{Q}_2^T \boldsymbol{M}\boldsymbol{Q}_2\right)\boldsymbol{Q}_1\right)^{-1} \boldsymbol{\Pi}_C - g\boldsymbol{Q}_5^T \boldsymbol{M}\boldsymbol{E}, \\ \dot{\boldsymbol{R}}_C &= \boldsymbol{R}_C \boldsymbol{S}\left(\left(\boldsymbol{Q}_1^T \left(\boldsymbol{J} + \boldsymbol{Q}_2^T \boldsymbol{M}\boldsymbol{Q}_2\right)\boldsymbol{Q}_1\right)^{-1} \boldsymbol{\Pi}_C\right) \quad . \end{aligned} \tag{44}$$

In Eq. (44), the kinematic part is a matrix, which can be changed as a vector type by Eq. (45):

$$\begin{aligned} &\dot{\boldsymbol{R}}_C{}^T \boldsymbol{E}_3 = \\ &= -\boldsymbol{S}\left(\left(\boldsymbol{Q}_1^T \left(\boldsymbol{J} + \boldsymbol{Q}_2^T \boldsymbol{M}\boldsymbol{Q}_2\right)\boldsymbol{Q}_1\right)^{-1} \boldsymbol{\Pi}_C\right) \boldsymbol{R}_C{}^T \boldsymbol{E}_3 . \end{aligned} \tag{45}$$

In Eq. (45), $\boldsymbol{E}_3 = [\boldsymbol{e}_3; \boldsymbol{e}_3; \boldsymbol{e}_3]$. All the parameters in rotation matrix are needed to be computed, so the total dimension of the dynamic equation is 30. The matrix's elements are computed directly, the structure conservation characters can clearly be represented according to the conservation elements in rotation matrix. The dynamic equation which uses rotation angle as parameter is as follows. In the kinematic relation, $\boldsymbol{\omega}_C$ is the first order derivation of $\boldsymbol{\theta}_C$, so the kinematic equation can be obtained by $\boldsymbol{\Pi}_C$ directly. The dynamics equation is as in Eq. (46). For the parameters are $\boldsymbol{\theta}_C$ and $\boldsymbol{\Pi}_C$, so the dimension of the dynamic equation is 6, which is much smaller than the former one.

$$\begin{aligned} &\dot{\boldsymbol{\Pi}}_C - \left(\boldsymbol{Q}_3^T \boldsymbol{J}\boldsymbol{Q}_1 + (\boldsymbol{Q}_4 + \boldsymbol{Q}_2\boldsymbol{Q}_3)^T \boldsymbol{M}\boldsymbol{Q}_2\boldsymbol{Q}_1\right)\dot{\boldsymbol{\theta}}_C + \\ &+ g\boldsymbol{Q}_5^T \boldsymbol{M}\boldsymbol{E} = 0, \\ &\boldsymbol{\Pi}_C - \left(\boldsymbol{Q}_1^T \left(\boldsymbol{J} + \boldsymbol{Q}_2^T \boldsymbol{M}\boldsymbol{Q}_2\right)\boldsymbol{Q}_1\right)\dot{\boldsymbol{\theta}}_C = 0. \end{aligned} \tag{46}$$

6. Simulation

The structure parameters of the system are as follows. the parameters of the system can be obtained by the measurement of the 3D model.

$m_1 = 0.87(\text{kg})$, $\boldsymbol{\rho}_1 = [-28.5 \quad -1 \quad -7.2]^T \times 10^{-3}(\text{m})$,
$l_1 = 78.5\times10^{-3}(\text{m})$, $J_{1x} = 6.55\times10^{-4}(\text{kg}\times\text{m}^2)$,
$J_{1y} = 7.10\times10^{-4}(\text{kg}\times\text{m}^2)$, $J_{1z} = 11.3\times10^{-4}(\text{kg}\times\text{m}^2)$,
$m_2 = 0.66(\text{kg})$, $\boldsymbol{\rho}_2 = [27 \quad -0.3 \quad 1.3]^T \times 10^{-3}(\text{m})$,
$l_2 = 81.5\times10^{-3}(\text{m})$, $J_{2x} = 1.79\times10^{-4}(\text{kg}\times\text{m}^2)$,
$J_{2y} = 4.45\times10^{-4}(\text{kg}\times\text{m}^2)$, $J_{2z} = 4.6\times10^{-4}(\text{kg}\times\text{m}^2)$,
$m_3 = 3.5(\text{kg})$, $\boldsymbol{\rho}_2 = [34 \quad -0.48 \quad 55]^T \times 10^{-3}(\text{m})$,
$J_{2x} = 1.46\times10^{-2}(\text{kg}\times\text{m}^2)$, $J_{2y} = 2.09\times10^{-2}(\text{kg}\times\text{m}^2)$,
$J_{2z} = 3.29\times10^{-2}(\text{kg}\times\text{m}^2)$.

The initial rotation angles of the system are as $\theta_{10} = \pi$ (rad), $\theta_{20} = 60°$, $\theta_{30} = 60°$, the initial angular velocities are $\omega_{10} = \pi$ (rad/s), $\omega_{20} = 60$ (°/s), $\omega_{30} = 60$ (°/s). Supping that there is no extra torque on the joint, the system can move free by the action of gravity. The simulation time is 10 s, the simulation results are as in Fig. 6 to Fig. 9.

The time variation of angular momentum is as in Fig. 6. According to the simulation results, the momentum which rotate along the x axis is conserved at 0 during the whole simulation. The variation of the other two momentums occurs an aperiodic change character, because the structure of the system is not symmetry in actual design process. The variation of rotation matrix $\boldsymbol{R}_z$ is as in Fig. 7. From the simulation results, the elements R_{13}, R_{23}, R_{31}, R_{32} are all keep 0 during the whole simulation, R_{33} keeps 1, R_{11} and R_{22} keeps the same regular, R_{12} and R_{21} have the different sign symbol. according to the above results, the structure of R_z is conserved. Similarly, the variation of $\boldsymbol{R}_y$ and $\boldsymbol{R}_x$ are as in

Fig. 8 and Fig. 9, and the simulation results also indicates the structure conservation of these two rotation matrixes.

The red curves in Fig. 6, Fig.7, Fig.8 and Fig. 9 are the simulation results of dynamic equation which use rotation angle as parameter. According to the simulation results, the two dynamics equation have the same results at beginning, and the simulation results occur obvious distinguish at 5 seconds. The simulation indicates that different geometry

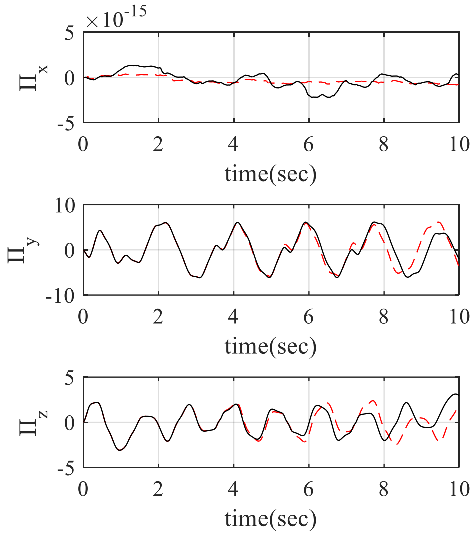

Fig. 6 The time variation of momentum

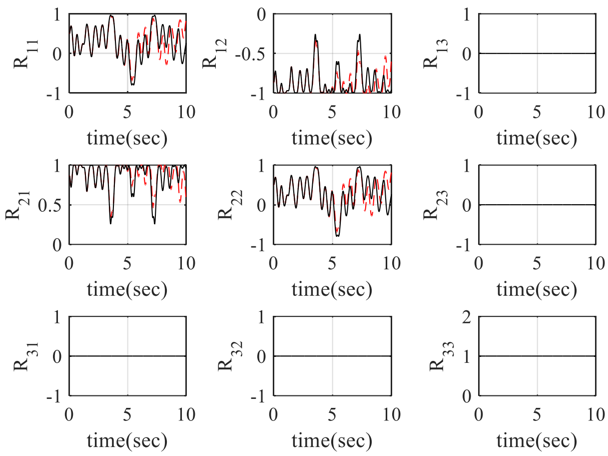

Fig. 7 The time variation of rotation matrix of $\boldsymbol{R}_z$

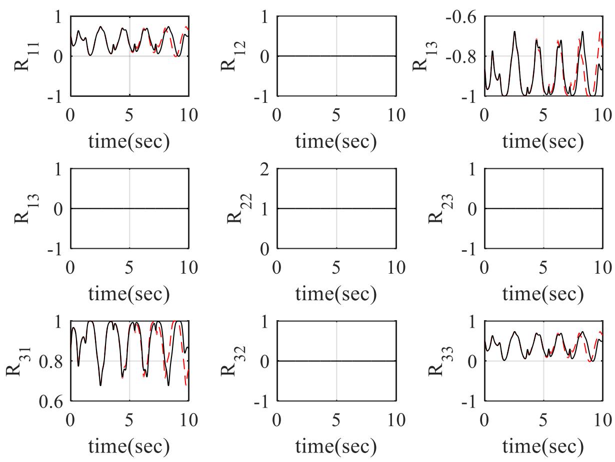

Fig. 8 The time variation of rotation matrix of $\boldsymbol{R}_y$

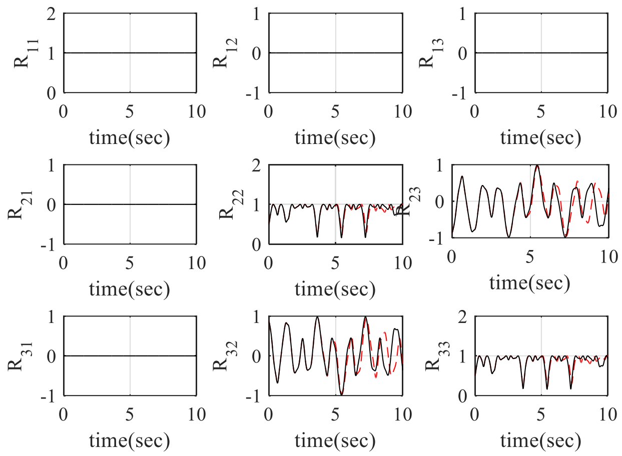

Fig. 9 The time variation of rotation matrix of $\boldsymbol{R}_x$

dynamics equation has different structure conservation character. Although the dimension of the geometry dynamics equation which using rotation matrix as parameter is much bigger, the structure of the system is much better conserved than the geometry dynamic equation which use rotation angular as parameter.

7. Conclusion

The 3D table with three different rotation coupling is a widely used model in the domain of spacecraft experiment, machine tool and robotics. It is also a typical model which can bridge the space rotation theory and engineer design together. In this exploration, the dynamics model of 3D table with three different rotation coupling is built with Lie group variational integrator method. The triangle transformation is avoided in the modelling process and the expression of dynamics equation change simplifier because the equation constitutes by matrix blocks. The simulation results indicate that the dynamics don't need special numerical solution algorithms because the geometry structure is conserved by the dynamics equation directly. It's a new dynamic equation which is friendly to engineer programming realization. This exploration also testify that two different dynamic model of 3D table lead to obvious different simulation under the same numerical algorithm. The most obvious character is that the simulation results occur obvious difference in 5 seconds, which is much smaller than 50 seconds in planar system (see reference [19, 20]). It means the different direction rotation coupling have a big influence on the structure conservation. This result testified that the geometry structure conservation character is influence both by the dynamics equation and numerical algorithms, so the conservation is not absolutely.

Declaration of conflicting interests

The authors declare no conflict of interest in preparing this article.

Data availability statement

No data were used to support this study.

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LIE GROUP VARIATIONAL INTEGRATOR FOR MULTI-BODY SYSTEM WITH ROTATION COUPLING IN SPACE

S u m m a r y

In multi-body system dynamics modelling, the body which rotate in space is complex and not easily to be expressed by Newton method, because the space rotation is realized by multi-joints rotation coupling with different direction. In this exploration, the Lie group variational integrator method is used to the dynamics modelling problem of multi-body system with three orthogonal direction joints coupling. Firstly, a mechanism accords with Cardan rotation regular is designed which can represent 3 different directions coupling, the kinematics model is derived out by matrix operation without triangle function. With inertia matrix and mass matrix of the multi-body system according to the topology structure, and the Lagrange function is built, and the dynamics equation is derived out with Lie group variational integrator method. With Legendre transformation, the Hamilton dynamics model is obtained. The differential computation of the momentum part is avoided, the scale of the dynamics model is greatly reduced. The Hamilton dynamics model with two different kinematics part are compared in simulation. The simulation results indicate that the different kinematic expression can lead to different structure conservation characters under same numerical computation method. This exploration offers a benefit attempt of using geometry method to dynamics modelling problem of tree structure multi-body system with different structure rotation matrixes coupling.

Keywords: rotation table, geometry dynamics, multi-body, numerical computation.

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