ISSN 1392 - 1207. MECHANIKA. 2007. Nr.6(68)

Stress distribution in multilayer structural element subjected to skew bending

J. Bareišis*, A. Daniškevičiūtė*, V. Kleiza*
**Kaunas University of Technology, Daukanto 12, 35209 Panevėžys, Lithuania, E-mail: tfdek@midi.ppf.ktu.lt*

1. Introduction

Multilayer structural elements (MSE) (including multilayer beams (MB)) have been recently used ever more widely in various industrial fields since they enable us to acquire structures with the necessary properties [1-5]. Strength, stiffness, and other characteristics of MB depend on mechanical characteristics of the material used and their arrangement in a structure as well as on geometrical parameters of components of the structural element [4-7]. In works [4, 5] the influence of various factors, such as elasticity modulus of material, number of layers and their arrangement in structural elements, symmetry with respect to one axis, on stiffness and strength of multilayer rods and beams has been considered. In [6, 7], some issues on geometric and stiffness centers, on variation of neutral layer directions and stiffness under bending have been discussed when the structure is asymmetric both in the sense of geometry and stiffness. In all these works multilayer beams are affected by pure bending. However, in real situations the cases of skew bending are rather frequent, and one of the main parameters for calculating such beams is their strength.

The target of this work is to present the methods for calculating stress of multilayer beams under skew bending within the limits of elasticity, to define stress values at the points typical of two-layer asymmetric beams and to consider the regularities of their variation dependent on the shape of beam cross-section, trajectory of its formation and on the values of elasticity modulus of materials that compose layers.

2. Mathematical model of a multilayer structural element

Assume MSE to be composed of n layers, elasticity modulus of the layers are $E_1, E_2, ..., E_n$, and the cross-sections occupy the simply connected domains K_i such that

$$K \subseteq K_{\square} = [0,1]\times[0,1],\ K = \bigcup_{i=1}^{n} K_i,\ K_i \cap K_j = \varnothing,\ i \neq j\,. \quad (1)$$

Then coordinates of stiffness center of MSE, directions of neutral layers and the values of extreme stiffness under bending can be expressed by inertia tensor, its characteristic directions and values. Axial stiffness density of MSE in this case can be defined by the function

$$E(x,y) = \sum_{i=1}^{n} E_i Ind_i(x,y) \quad (2)$$

where $Ind_i(x,y) = \begin{cases} 0, & (x,y) \notin K_i \\ 1, & (x,y) \in K_i \end{cases}$ is the indicator function of the set.

Let us assume $\boldsymbol{E} = (E_1, E_2, ..., E_n)$, then, with respect to (2)

$$m_{pq}(\boldsymbol{E}) = \iint_K x^p y^q E(x,y)dxdy \quad (3)$$

by which we express the normal stress appearing in the cross-section of MB. The formulas for calculating the coordinates of stiffness center as well that of axial and flexural stiffness have been obtained in [7].

If MB cross-section is affected by the bending moment $\boldsymbol{M}$ noncollinear to the main inertia tensor directions (skew bending) and the trace of its action plane crosses the stiffness center, then the normal stress at each MB cross-section point $P(x,y) \in K$ is

$$\sigma(x,y) = E(x,y)\left(\boldsymbol{M},\ r(x,y)\right) \quad (4)$$

here x and y are the coordinates of point P in the global coordinate system (GCS), $M_{x_{cs}}$ and $M_{y_{cs}}$ are the components of the bending moment vector $\boldsymbol{M} = \begin{pmatrix} M_{x_{cs}} & M_{y_{cs}} \end{pmatrix}$ in the central principal coordinate system (CPCS),

$$r(x,y) = \left(x_{cp}/m_{02}(\boldsymbol{E}) \quad y_{cp}/m_{20}(\boldsymbol{E}) \right)$$

where x_{cp} and y_{cp} are coordinates of the point (x,y) in CPCS, $J_1 = m_{20}(\boldsymbol{E})$ and $J_2 = m_{02}(\boldsymbol{E})$ are modulus weighted inertia moments (3) with respect to the axes of CPCS.

Expression (4) completely describes the scalar field of normal stress $\sigma(x,y),\ (x,y) \in K$ and allows us to define its structure. Let α be an angle between GCS and CPCS axes, and θ be an angle between the axis x of CPCS and the bending moment vector $\boldsymbol{M}$ (Fig.1). Then, taking into consideration that the gradient of the scalar field of normal stress is constant in each domain K_i and is equal

$$grad_{K_i}\sigma(x,y) = ME_i\left(\frac{cos\theta tan\alpha}{J_1\sqrt{1+tan^2\alpha}} + \frac{sin\theta cot\alpha}{J_2\sqrt{1+cot^2\alpha}}\right)\boldsymbol{i} + ME_i\left(\frac{-cos\theta}{J_1\sqrt{1+tan^2\alpha}} + \frac{sin\theta}{J_2\sqrt{1+cot^2\alpha}}\right)\boldsymbol{j}$$

We conclude that the level lines of the scalar field of normal stress are straight lines in each domain K_i (i.e., within the limits of each layer), and the function $\sigma(x,y)$ is piecewise linear in the domain K (only finite discontinuities in the contours of domains K_i are possible).

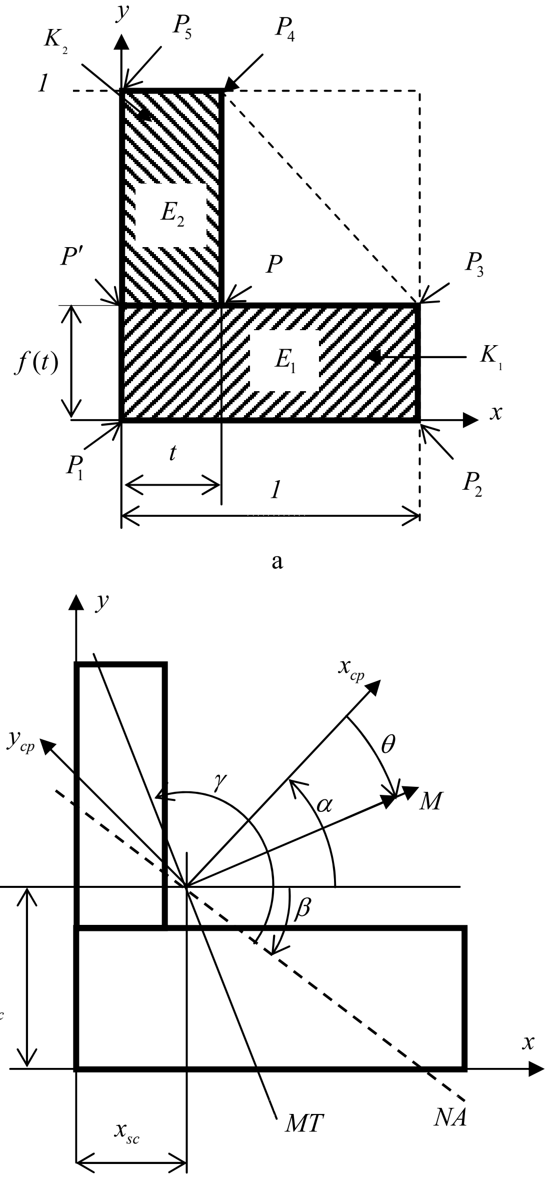


Fig. 1 Geometry of a structural element: a - global $\{x, y\}$ and principal stiffness coordinate systems. $P_1 - P_5$ vertexes of the convex hull of cross section *K*; b - *NA* is neutral position axis and *MT* is the trace of bending moment acting plane; x_{sc} and y_{sc} are coordinates of stiffnes center

3. Object of study

The MSE subjected to bending are often formed of rectangular shape cross-section layers, the dimensions generally are not uniform and the cross-section of MSE does not possess a single inverse axis of symmetry. Moreover, for formation of the layers the materials of different elasticity modulus E_i are employed, therefore the structure can be asymmetric not only in geometric sense but also in

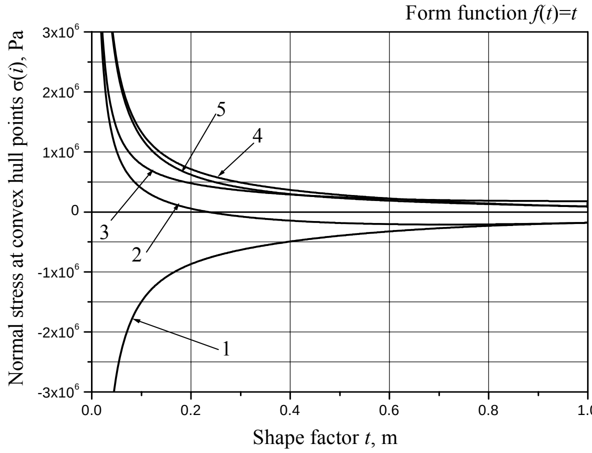


Fig. 2 Normal stress at convex hull points. Curve number corresponds to vertex number of the cross section convex hull. The ratio of elasticity modulus E_2/E_1=0.5

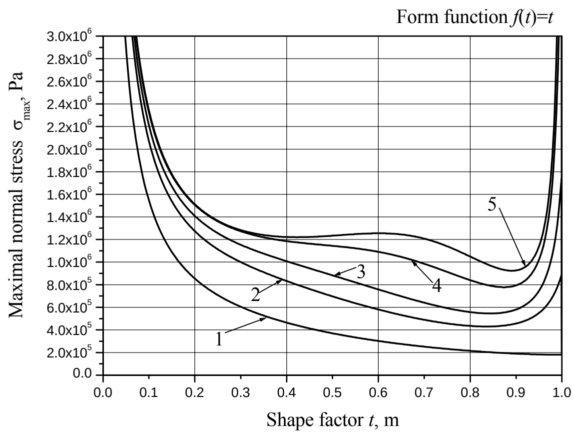
the sense of stiffness, and the stiffness center generally can not coincide with the geometric one. Such a structure is a two layer ($E_1 \neq E_2$) composite formed from two rectangles with a mutual share of the contour (Fig. 1, a, segment $P'P$). In [8] the dynamics of values variations of a two-layer beam geometric and stiffness centers and that of neutral layers directions and variations of extreme stiffness values at bending when the structural element was formed by moving point *P* along diagonal of the square (Fig. 1, a) was investigated. In this study, the investigation results obtained at structural element formation at point *P* moving along curves laying in a unit square 1×1 m (Fig. 1, a), are defined by function $f(t)$. Thus, the object under study – a two-layer structure satisfies the condition (1) and

$$n = 2,\ K_1 = [0,1]\times[0, f(t)]$$
$$K_2 = [0,t]\times[f(t),1],\ t \in [0,1]$$
$$P_1 = (0,0),\ P_2 = (1,0),\ P_3 = (1, f(t)),\ P(t, f(t))$$
$$P_4 = (t,1),\ P_5 = (t,1),\ P' = (0, f(t))$$

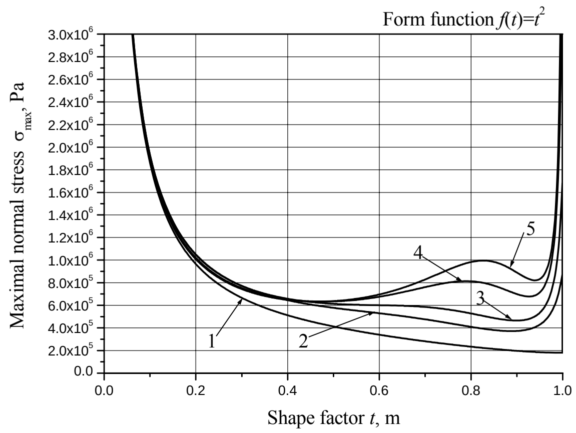
here $P_1, P_2, P_3, P, P_4, P_5$ are vertexes of MSE cross section K_1 and P_1, P_2, P_3, P_4, P_5 are vertexes of the convex hull of cross section K_2, $f(t)$ is a continuous function satisfying condition $0 \le f(t) \le 1,\ t \in [0, 1]$. The shape of MSE cross-section depends on $f(t)$ therefore $f(t)$ further is called as form function and *t* is called a shape factor. A part of the investigation was performed at $f(t) = t^m,\ m = 1,\ 2,\ 0.5$. The angles $\alpha, \beta, \lambda, \theta$ are defined in Fig, 1, b.

4. Investigation results

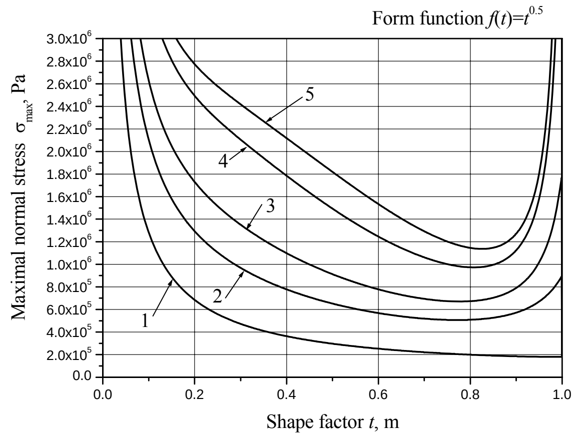
This work presents the results of stress investigation when the structural elements is formed with the point *P* moving along the curves $f(t) = t^m$, index *m* of the cross-section shape being equal to 1, 2 and 0.5. In all cases, the bending moment vector *M* crosses the stiffness center and is perpendicular to the axis *y* of the global coordinates system ($\theta = -\alpha$). With geometry the variation of cross-section layers, the stiffness center coordinates x_{sc}, y_{sc} and



a



b



c

Fig. 3 Dependencies of maximal normal stresses on form function: a - $f(t)=t$, b - $f(t)=t^2$, c - $f(t)=t^{0.5}$. The ratio of modulus E_2/E_1: 1 - (curve *1*); 5 - (curve *2*); 10 - (curve *3*); 30 - (curve *4*); 55 - (curve *5*)

the angle α between GCS and SPCS axes were calculated in each case.In the case $\theta = -\alpha$ under consideration, the trace of bending moment action plane (Fig. 1, b, straight line *MT*) is perpendicular to the axis *x*. In case of skew bending, the angle between neutral axis (Fig. 1, b, straight line *NA*) and that of *x* is denoted as β, while the angle between neutral axis and the trace of moment action plane is denoted as γ.

The first parameters (stiffness center, position of the principal inertia moment axes) have been calculated using the mathematical model [6]. The ratios of elasticity modulus of the layer materials are analogous [6, 7], $E_2/E_1 = 0.5, 1.0, 5.0, 10, 30, 50$, where $E_1 = 3000$ MPa (this corresponds polycarbonate elasticity modulus, and under maximal ratio of the modulus, we obtain the modulus close to that of steel elasticity modulus). Stress field was calculated using the mathematical model proposed (4), the bending moment vector modulus being $|M| = 3 \cdot 10^4$ Nm.

Fig. 2 presents stress variation at the edge points of the beam cross-section convex hull (since only at them the maximum of the absolute stress value can be reached), where the ratio of elasticity modulus of materials that compose the layers is $E_2/E_1 = 0.5$, and the beam shape varies along the diagonal of unit square, i. e., index of the form function $f(t) = t^m$ is *m*=1. We have found that with an increase of the form shape factor *t*, stress decrease at all edge points of the beam cross-section convex hull along the curves close to exponent, because the cross-section area and the beam stiffness increase. The beam stiffness increases if the ratio of elasticity modulus of the layers considered is ($E_2/E_1 = 0.5$) [6, 7]. The highest stress values have been obtained at point P_1 (Fig. 2, curve *1*) that belongs to the layer whose elasticity modulus is higher. The stress at point P_4, whose distance to the neutral axis is the greatest, is 20% lower than at point P_1. Note that under the action of a bending moment with $\theta = -\alpha$ up to the value $t = 0.23$, the compressing stress is acting only at one point (Fig. 1, point P_1). This is due to the position of stiffness centre (it is shifted downwards and to the left) as well as to leaning angle β of the neutral axis leading via the stiffness center. The angle β is varying (with $E_2/E_1 = 0.5$ and *m*=1) from $-27°$ as up to $-22°$ degrees as $t = 0.23$, and as $t = 1.0$ is decreasing to zero. With an increase of parameter *t* values, the value of angle β is decreasing from maximal negative value, which is fluctuating from $-37°$ up to -4° degrees depending on the cross-section form index *m*, to zero. The lowest value of angle β has been obtained as *m*=0.5. With such a cross-section form index, the value of angel β increases up to zero (as $t = 1.0$).

When increasing the elasticity modulus of the second layer material, the stress at cross-section point P_4 (Fig. 1, a), with a lower elasticity modulus in the layer are decreasing and approach zero, since the influence of the first layer on the beam stiffness is decreasing. The highest stress values are obtained at the second layer point P_4 the distance of which to the neutral axis is the largest. At cross-section point P_4 (Fig. 1, a), in which stress values are the highest, the variation of these values on the ratio of elasticity modulus and the trajectory of cross-section formation are presented in Fig. 3. By comparing the obtained dependences of stress variation, we can see that their nature depends on the cross-section formation trajectory, i.e., on the index *m* cross-section form. That is natural because under a same abscissa of point *P*, geometric parameters dependent on the shape factor of the layers, composing cross-section, are different and stiffness of the structural element is also different thereby. It is noteworthy that with an increase in the ratio of elasticity modulus, say 10 times, as $t = 0.4$, the maximal stress (at point P_4) increases 1.5-

2.2 times (Fig. 3, a, curves *1*, *3*, *2* and *5*), i. e., as $f(t)=t$, and 2.7-2.9 times as $f(t)=\sqrt{t}$ (Fig. 3, c, curves *1, 3, 2* and *5*). Meanwhile, as $f(t)=t^2$, the value of the parameter *t* being the same, there are no differences between stress values (Fig. 3, b).

It has been established that absolute stress values also differ considerably, if the index *m* values of the beam cross-section shape are different, for instance, if $m=1.0, t=0.4$, $E_2/E_1=50$ stress at the beam cross-section point P_4 are equal to 1.2 MPa (Fig. 3, a, curve *5*), and if $m=2$, stress are twofold lower: 0.63 MPa (Fig. 3, b, curve *5*), and if $m=0.5$, stress are equal to 2.1 MPa (Fig. 3, c, curve *5*).

When forming beam cross-section according to $f(t)=\sqrt{t}$, the amount of material with higher stiffness is smaller than that of the material of lower stiffness. Therefore, in this case, beam stiffness is considerably lower than that when the beam cross-section is formed according to $f(t)=t^2$.

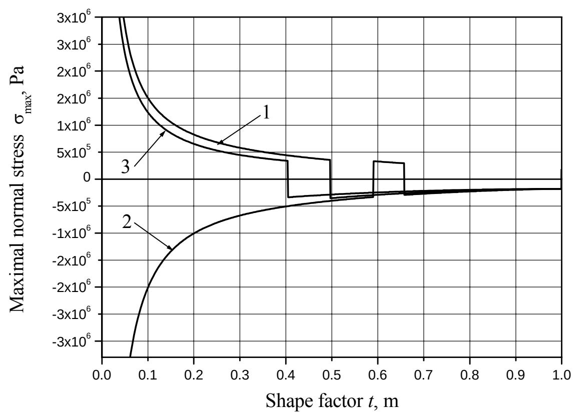


Fig. 4 Dependencies of maximal normal stresses on form function: $f(t)=t$, (curve *1*); $f(t)=t^2$, (curve *2*); $f(t)=t^{0.5}$, (curve 3). The ratio of modulus E_2/E_1=0.877

Note that with an increase of the ratio between elasticity modulus of materials that compose the layers, the nature of dependences of stress variation on the parameter *t* changes. In the case of a homogeneous beam, with an increase on the parameter *t*, we obtain exponentially decreasing stress values (Fig. 3, a, b, c, curve *1*).

Meanwhile, in the case of a two-layer beam under skew bending, one can notice a stress decrease in the values of the parameter *t* up to $0.8 \div 0.9$ and an increase in higher *t* values (Fig. 3, a, b, c, curves *2-5*). The intensity of stress decrease is higher with a lower ratio of elasticity modulus. When formation of the beam cross-section is going on according to $f(t)=t^2$ and the ratio of layer elasticity modulus is equal to 50, we can distinguish three stress variation intervals in the stress decreasing stage (Fig. 3, a), if *t* is varying from 0.05 up to 0.3, we have a rapid stress decrease, if $t=0.3-0.7$, stress is actually constant at the considered point P_4, and if $t=0.7-0.85$, stress is intensively decreasing again. This kind of stress variation can partly be explained by a complicated variation of stress under bending about the main axes. In [7], it has been defined that in the case of a beam considered with the ratio 50 of elasticity modulus, D_{max} (maximal bending stiffness) has two explicitly expressed maxima with the minimal D_{max} value, if $t=0.5$. At this value of the parameter *t*, D_{max} acquires maximum that is about 35% lower than the minimal D_{max} value [7]. Thus, we can state that stress variation dependences are not inversely proportional to bending stiffness variation under bending.

When considering stress variations in a two-layer asymmetric, with respect to both axes, beam under skew bending we have noticed that not always the maximal stress is attained in the layer with the highest stiffness. Fig. 4 illustrates the curves of maximal stress variation where the ratio of elasticity modulus of materials that compose the layers is lower than a unit ($E_2/E_1=0.877$). A jump of stress denotes the moment when the maximal stress jumps from one point of a convex hull of the beam cross-section to another. For instance, if formation of the beam cross-section is going on along the straight line $f(t)=t$, for $t<0.5$, the highest absolute stress is at point P_4 the most distant from neutral axis, and for $t \geq 0.5$ the maximal absolute stress is attained in the material of greater stiffness (Fig. 4, curve *1*). It is of interest that a stress jump occurs only under certain ratios between elasticity modulus of the materials that compose layers. Limit values of elasticity modulus ratios, when maximal stress transition is observed from one point of cross-section to another, depend on the trajectory of cross-section formation. If formation of the beam cross-section is going on along the straight line $f(t)=t$, a stress jump is observed as the ratio E_2/E_1 of elasticity modulus is varying form 0.526 to 0.995, $t=0.05$ and $t=0.995$. If the cross-section formation is going on along the curve $f(t)=t^2$, then maximal stress changes its place even twice (Fig. 4, curve *2*). Limit values of the ratio between elasticity modulus are from 0.867 to 0.995. If the ratio of elasticity modulus is lower than 0.867, then maximal stress in the whole interval of the parameter *t* variation is obtained at point P_1, and if the values are a little higher than the lower limit value, then the loop width is low (Fig. 4, curve *2*) and it increases until the maximal stress is reached only at point P_4 with E_2/E_1>0.995. And finally, if the beam cross-section formation takes place along the curve $f(t)=\sqrt{t}$, then the stress jump occurs as the ratio E_2/E_1 is varying in a very wide interval of the parameter *t* variation (from 0.1 to 0.995) (Fig. 4, curve *3*).

Thus, we have established that maximal stress can arise not only in the material with higher elasticity modulus, and that the rise of maximal stress in the material of another layer depends not only on elasticity modulus ratios, but also on the cross-section shape index *m*, i. e., on the cross-section shape function.

5. Conclusions

1. A mathematical model of multilayer beams subjected to skew bending as well as the methods for calculating stress that are very convenient to calculate beam stress of any cross-section configuration has been proposed.

2. It was established that by changing the layer geometry of two-layer beam cross-section, stress at all the cross-section vertex point is varying nonuniformly. At a

part of points stress decreases along the curve close to the cubic parabola, while at the points most distant from neutral layer, stress variation is of a complicated nature, i. e., it has one or even two stress minima.

3. In many cases, if the ratios between elasticity modulus of materials that compose the layers exceed a unit, the maximal stress is attained at the point of a higher rigidity layer that is the most distant from the neutral layer.

4. It was defined that if the ratios between elasticity modulus of materials that compose the layers are lower than a unit, maximal stress can be obtained either in one or another material and this change can occur once or twice, dependent on the layer geometry. Thus means that it can not be know beforehand in which layer of the structure maximal stress is, what can lead to the loss of beam strength.

5. The limits for the parameter *t* variation under which the transition of maximal stress to another layer takes place as have been determined well as the fact that they depend on the trajectory of cross-section formation.

References

1. **Bulavs, F., Radinsh, I., Tirans, N.** Improvement of capacity in bending by the use FRP layers on RC beams.-J. of Civil Engineering and Management, 2005, No.3(11), p.169-174.
2. **Jaras, A., Kačianauskas, R.** Deflection analysis of bisteel I section beams with elastic-perfectly plastic behaviour of the web.-Mechanika. -Kaunas: Technologija, 2003, Nr.5(43), p.5-10.
3. **Gurkšnys, K., Kvedaras, A., Kavaliauskas, S.** Behaviour evaluation of "sleeved" connectors in composite timber - concrete floors.-J. of Civil Engineering and Management, 2005, No.4(11), p.277-282.
4. **Bareišis, J.** Stiffness and strength of multilayer beams. -J. of Composite Material, 2006, No.6(40), p.515-531.
6. **Garuckas, D., Bareišis, J.** The influence of different factors on the stiffness and strength of multilayered composite elements.-Mechanics of Composite Materials, 2003, No.2(39), p.500-511.
7. **Bareisis, J., Kleiza, V., Kleiza, J.** Investigation of the flexural stiffness of asymmetric multilayer beams. -Strength of Materials, 2006, No.6(38), p.601-612.
8. **Bareišis, J., Kleiza, V.** Stiffness center and neutral layer direction investigation method and its application to asymmetric multilayer structural elements. -Mechanika. -Kaunas: Technologija, 2004, Nr.4(48), p.5-12.

J. Bareišis, A. Daniškevišiūtė, V. Kleiza

ĮTEMPIŲ PASIKIRSTYMAS ĮSTRIŽAI LENKIAMAME DAUGIASLUOKSNIAME KONSTRUKCINIAME ELEMENTE

R e z i u m ė

Pateikti įstrižai lenkiamų daugiasluoksnių sijų, turinčių geometrinę ir/ar standuminę asimetriją, stiprumo ir standumo tyrimų rezultatai. Pasiūlytas normalinių įtempių (stiprumo) ir standumo lenkiant bet kuriame daugiasluoksnės sijos skerspjūvio taške skaičiavimo matematinis modulis. Ištirta standumo lenkiant ir stiprumo kitimo priklausomybių kinetika, kintant skerspjūvio geometriniams parametrams ir sijos sluoksnių tamprumo modulių santykiams. Nustatyta, kad įstrižai lenkiamos daugiasluoksnės sijos stiprumas, pagrindinai, priklauso nuo standumo centro padėties ir neutraliosios plokštumos padėties.

J. Bareišis, A. Daniškevičiūtė, V. Kleiza

STRESS DISTRIBUTION IN MULTILAYER STRUCTURAL ELEMENT SUBJECTED TO SKEW BENDING

S u m m a r y

Results of researches on strength and stiffness of diagonally bending sandwich girders having geometric and/or stiffness asymmetry have been given. A mathematical module for calculation of normal strain (strength) and stiffness on bending at any cross-section point of sandwich girder has been proposed. The kinetics of stiffness on bending and strength variation dependences when geometric parameters of a cross-section and tension modulus ratios of girder layers are changing have been examined. It has been established that the strength of a diagonally bending sandwich girder on the whole depends on the position of the centre of stiffness and the position of the neutral plane.

Й. Барейшис, А. Данишкевичюте, В. Клейза

РАСПРЕДЕЛЕНИЕ НАПРЯЖЕНИЙ В КОНСТРУКТИВНОМ ЭЛЕМЕНТЕ ПРИ КОСОМ ИЗГИБЕ

Р е з ю м е

Представлены результаты исследования прочности и жесткости многослойных балок, имеющих геометрическую и (или) жесткостную асимметрию, при косом изгибе. Предложена математическая модель для расчета изгибной жесткости и нормальных напряжений (прочности) в любой точке поперечного сечения многослойной балки. Изучена кинетика изгибной жесткости и прочности в зависимости от изменения геометрических параметров сечения, а также отношения модулей упругости слоев. Установлено, что прочность многослойных балок при косом изгибе во многом зависит от положений центра жесткости и нейтральной плоскости.

Received October 23, 2007