

Reliability-Based Proportional Topology Optimisation of Continuum Structures with Smooth Boundaries

Lingyun YANG^{*,**}, Yong SONG^{**}, Xiaofeng HU^{**}, Xiong RAO^{***,****}, Run DU^{*,****}, Wenming CHENG^{*,****}

^{*}*School of Mechanical Engineering, Southwest Jiaotong University, Chengdu 610031, China*

^{**}*Chengdu Fifth Construction Engineering Co., Ltd. of CDCEG, Chengdu 610031, China*

^{***}*School of Mechanical Engineering, Southwest Jiaotong University, Chengdu 610031, China,*

E-mail: x_rao@my.swjtu.edu.cn (Corresponding Author)

^{****}*School of Mechanical and Electrical Engineering, Guang'an Institute of Technology, Guang'an 638000, China*

^{*****}*Technology and Equipment of Rail Transit Operation and Maintenance Key Laboratory of Sichuan Province, Chengdu 610031, China*

<https://doi.org/10.5755/j02.mech.44172>

1. Introduction

Uncertainty is an inherent property of nature, and this characteristic extends deeply into the field of structural design. It is specifically reflected in core design parameters such as structural geometric dimensions and loads acting on structures, all of which exhibit significant uncertain features. Existing studies have confirmed that incorporating the uncertainty of design parameters into structural topology optimisation (TO) can not only adjust the structure's force transmission paths but also generate diversified, optimised structural forms with distinct shapes [1-4]. As a key branch in the field of structural optimisation design, reliability-based TO (RBTO) serves as a core technical solution for addressing structural optimisation challenges under the prerequisite of "uncertain design parameters". By integrating TO methods with reliability analysis techniques to conduct structural optimisation design, this technology quantitatively incorporates the influence of various uncertainty factors. Consequently, the finally formed optimised structure not only has higher adaptability to the practical application needs of engineering but also can achieve an optimal balance between structural safety performance and economic costs [5-7]. For this reason, RBTO has become a key research direction within the scope of structural optimisation design and has gained extensive attention and high recognition from the academic community.

Regarding the RBTO of continuum structures accounting for uncertainties in design parameters, relevant scholars have conducted extensive research from various perspectives. Zhan and Bai [8] put forward a non-probabilistic RBTO method that addresses uncertainties associated with distributed loading. To substantiate the validity of the presented approach, several numerical examples that involve uncertain loading fields and cover both 2D and 3D structures are provided. To strike an ideal balance among reliability, robustness, cost, and structural safety, Yang et al. [9] pioneeringly proposed a unified robust RBTO methodology for continuum structures with loading uncertainty. Hu et al. [10] proposed an innovative decoupled framework for RBTO that incorporated the local radial point interpolation method and accounted for stress and displacement constraints, achieving optimal material distribution while satis-

fying strength and stiffness constraints. Li et al. [11] put forward a level set RBTO method that accounts for multi-source interval uncertainties and sensitivity mapping, and this approach has been successfully utilised in the TO of the rocket skid structure. Zhan et al. [12] offered a geometrically nonlinear non-probabilistic RBTO method specifically designed to address non-probabilistic load parameters as well as geometric field uncertainties. Zhan et al. [13] proposed a fatigue RBTO framework for compliant mechanisms that incorporates material uncertainties; in contrast to the outcomes of deterministic TO, the compliant mechanisms derived from the proposed framework demonstrate superior fatigue reliability. It has been demonstrated in published literature that RBTO problems for continuum structures can be efficiently resolved using TO techniques such as the variable density method, level set method (LSM), and evolutionary structural optimisation method.

The proportional TO (PTO) algorithm, which belongs to the category of variable density methods [14], is a novel heuristic algorithm presented by Biyikli and To [15]. The fundamental concept of this approach is to allocate material based on the proportion of each element's stress or compliance values relative to the total stress or total compliance values of all structural elements. Owing to its advantages of dispensing with the requirement to acquire sensitivity information, conceptual simplicity, and ease of implementation, it is gradually attracting increasing attention from scholars in the relevant academic field. Fu et al. [16] put forward a strategy for addressing the TO problem of trusses exhibiting low-level nodal location uncertainty by means of the PTO approach. Test cases involving multiple types of trusses were utilised to validate the method's excellent efficiency. Cui et al. [17] offered a multi-material PTO method built upon a revised density interpolation strategy employing the S-shaped logistic function, and have accordingly obtained topological structures featuring distinct boundaries. Nguyen et al. [18] have extended the PTO algorithm to enable its application in two-scale concurrent TO, where both the structural configuration and the material's cellular microstructure are incorporated into the design process. Vo et al. [19] put forward an iso-geometric PTO approach (termed IGA-PTO), and by leveraging this approach, successfully obtained topological structures with smooth,

well-defined boundaries. The IGA-PTO approach was subsequently applied to solving multi-material TO problems [20]. Kumkam and Slesongsom [21] proposed an efficient method for RBTO that addresses uncertainty and utilises PTO to obtain structures with optimal reliability. Rao et al. [22] developed a novel PTO approach integrating level set function (LSF) and evolutionary strategy for addressing the 2D and 3D TO problems, and subsequently employed this approach in designing microstructures for porous materials with superior mechanical properties, thereby yielding microstructural topologies characterised by well-defined interfaces and smooth boundaries [23]. It is not difficult to find from the aforementioned published literature that scholars have taken the PTO method as a core framework to carry out solution work for various structural TO problems, covering the direct application and improved expansion of this method. However, research reports on solving structural RBTO problems using PTO-type methods remain relatively rare. Although some works [21,24] have achieved the solution goal of structural RBTO problems via the PTO method, the resulting topological structures exhibit grey areas or serrated boundaries, a drawback that necessitates extensive post-processing prior to practical engineering implementation.

In this paper, we propose a reliability-based PTO (RPTO) method to address continuum structural TO problems. Specifically, a probabilistic framework is adopted to characterise uncertainties in design variables, and a corresponding RBTO model for continuum structures is set up to minimise structural compliance under dual constraints of probabilistic reliability and material volume. To ensure smooth boundaries of optimised structures, the RPTO method incorporates an LSF. The effectiveness of the proposed RPTO method is verified through several typical test cases involving 2D and 3D TO problems.

2. Proportional Topology Optimisation Algorithm Integrating the LSF

In the deterministic TO phase of implementing the RPTO method, referring to the published paper [22], the PTO algorithm integrating the level set function (PTO-LS) is used here. PTO-LS method has the advantages of not requiring sensitivity calculations and the optimised structure having smooth boundaries. This method includes the following main contents:

2.1. Compliance minimisation problem

Referring to the literature [22, 25], the mathematical model for the minimisation of structural compliance subject to material volume constraint employed here can be stated as:

$$\begin{cases} \min : C = \mathbf{U}^T \mathbf{K} \mathbf{U} \\ s.t.: \begin{cases} \mathbf{K} \mathbf{U} = \mathbf{F} \\ \sum_{m=1}^N \rho_m v_m - V_f = 0 \\ 0 \leq \rho_m \leq 1, (m = 1, 2, \dots, N) \end{cases} \end{cases}, \quad (1)$$

where C denotes compliance; $\mathbf{U}$ is the displacement vector;

$\mathbf{F}$ indicates the external load; $\mathbf{K}$ represents the global stiffness matrix; ρ_m is the pseudo density (i.e., design variable); N is the total quantity of elements; v_m denotes elemental volume; V_f indicates the material volume fraction.

The Young's modulus $E(\rho_m)$ can be acquired via the following formula:

$$E(\rho_m) = K_v + (K_s - K_v) \cdot \rho_m^p, \quad \rho_m \in [0, 1], \quad (2)$$

where K_v and K_s denote the Young's moduli assigned to the void and solid elements, respectively, and p denotes a penalty coefficient.

2.2. Compliance proportion calculation and filtering

In this optimisation framework, after computing the compliance value for each element via finite element analysis (FEA) is completed, the compliance proportion value of each element can be calculated by:

$$C_m^p = \frac{C_m}{\sum_{m=1}^N C_m v_m}, \quad (3)$$

where, besides the nomenclature provided for Eq. (1), C_m and C_m^p indicate the compliance value and compliance proportion value of element m , respectively.

After obtaining the compliance proportion value associated with each element inside the structure, the compliance proportion filtering technique [22] is used here to reduce local fine structures to better adapt to engineering applications. The mathematical representation of this filtering scheme is given below:

$$C_m^{pf} = \frac{\sum_{j=1}^{N_f} \beta_{m,j} C_j^p}{\sum_{j=1}^{N_f} \beta_{m,j}}, \quad (4)$$

in which C_j^p denotes the compliance proportion value corresponding to element j within the filter radius r_f before filtering, C_m^{pf} stands for the compliance proportion value relative to element m in the structure following the implementation of the filtering scheme, N_f represents the overall count of elements inside the radius r_f , and $\beta_{m,j}$ indicates the weighting factor, which can be obtained through the following equation:

$$\beta_{m,j} = \begin{cases} \frac{r_f - D(m,j)}{r_f}, & \text{for } D(m,j) < r_f, \\ 0, & \text{for } D(m,j) \geq r_f \end{cases}, \quad (5)$$

in which $D(m,j)$ stands for the distance separating the centres of elements m and j .

2.3. Nodal compliance proportion

To facilitate obtaining smooth and distinct boundaries, the design scheme of the LSF founded on the nodal compliance proportion was used in this paper [22]. Here,

the nodal compliance proportion C_k^{NCP} is defined as the average value of the compliance proportion values of all elements sharing node k in common, which can be expressed as:

$$C_k^{NCP} = \frac{1}{N_e} \sum_{e=1}^{N_e} C_e^{pf}, \quad (6)$$

in which N_e represents the overall count of elements linked to the shared node. Concerning bi-linear four-node quadrilateral elements, N_e commonly takes three values: 1, 2, and 4. To elaborate, a node is identified as a corner node (at the structure's corners), a boundary node (on the structure's edges), or an inner node (inside the structure) when N_e is 1, 2, or 4, respectively [26]. And C_e^{pf} indicates the element's compliance proportion value calculated using Eq. (4).

For the purpose of enhancing the algorithm's convergence performance, an additional averaging step is applied to each node's compliance proportion value:

$$C_k^{NCP(n)} \leftarrow \frac{C_k^{NCP(n)} + C_k^{NCP(n-1)}}{2}, \quad (7)$$

in which n represents the iteration step. The terms $C_k^{NCP(n)}$ and $C_k^{NCP(n-1)}$ refer to the compliance proportion value that corresponds to node k at iteration n and iteration $n-1$, respectively.

2.4. Description of LSM

Leveraging the LSM's intrinsic capabilities in generating smooth, well-defined iso-contours and iso-surfaces, this study integrates it into the proposed optimisation framework. In the LSM, the LSF ψ is employed to characterise the topological configuration. A mathematical formulation describing the domain's boundaries and additional related parts can be formulated:

$$\begin{cases} \psi(\gamma, t) > 0, & \forall \gamma \in \Omega \setminus \Gamma \\ \psi(\gamma, t) = 0, & \forall \gamma \in \Gamma \\ \psi(\gamma, t) < 0, & \forall \gamma \in D \setminus (\Omega \cup \Gamma) \end{cases}, \quad (8)$$

where $\gamma \in R^2$ or R^3 denotes a point within the structure $D \in R^2$ or R^3 , Ω indicates the solid domain with Γ as its boundary, and t stands for the pseudo-temporal variable employed to characterise the update iterations during the LSF's evolutionary progression.

To prepare the structure for FEA, the LSF is transformed into a physical representation. This transformation is accomplished via an appropriately defined mapping function $M(\psi)$, which assigns the LSF values to the design prototype's outer surface as below:

$$M(\psi) = \begin{cases} 1, & \text{if } \psi \geq 0 \\ 0, & \text{if } \psi < 0 \end{cases}. \quad (9)$$

For computational efficiency in FEA, the ersatz material model [27] is utilised. This model allows for a fixed mesh established at the optimisation outset, requiring no updates later. In this implementation, the LSF-based ersatz material is represented by a density variable ρ_m , calculated via

the Heaviside mapping $M(\psi)$:

$$\rho_m = \frac{\int_{D_m} M(\psi) d\Omega}{\int_{D_m} d\Omega}, \quad (10)$$

where D_m defines the element domain. In actual computation, ρ_m is determined by the subsequent expression:

$$\rho_m = \frac{A_s}{A_m}, \quad (11)$$

in which A_s corresponds to the solid area (2D) or solid volume (3D), and A_m refers to the element area (2D) or element volume (3D). Once element density ρ_m is computed, Young's modulus $E(\rho_m)$ is derivable via Eq. (2).

2.5. Update strategy

Level-set value computation requires predefined goal material volume in the present iteration. The methodology incorporates the evolutionary strategy [22, 28], implemented as follows:

$$V^{(n)} = \begin{cases} \min\{V_{am}, V^{(n-1)}(1+er)\}, & \text{for } V_{am} \geq V_{sgd} \\ \max\{V_{am}, V^{(n-1)}(1-er)\}, & \text{for } V_{am} < V_{sgd} \end{cases}, \quad (12)$$

where er stands for an evolution rate, which is used to regulate the volume of material added or subtracted during each optimisation iteration, until the allowable material volume V_{am} is achieved. $V^{(n)}$ denotes the goal volume for the present iteration, and V_{sgd} indicates the starting guessed design volume (for example, 1 representing the complete material state). The algorithm maintains a constant material volume and only alters the structural layout once V_{am} is attained.

In this approach, an ordinary differential equation [26] serves as the theoretical foundation for deriving the LSF update strategy. When integrating this strategy with the optimality criterion (OC) equation (which is built on the Karush-Kuhn-Tucker (KKT) condition), nodal compliance proportion ($C_k^{NCP(n)}$) in Eq. (7) is used to control the update of LSF:

$$\frac{d\psi}{dt} - C_k^{NCP(n)} = 0. \quad (13)$$

Need to add that Eq. (13) is constructed based on any point in the structure; this scheme can be used to analyse all mesh nodes. Following the incorporation of the nodal compliance proportion, the LSF evolution becomes applicable to structural optimisation. For the OC method, it can be readily demonstrated that the OC applicable to elements is equivalent to that for nodes [26]. Consequently, node-based OC can be replaced with the element-based one. The node-based OC in this case is given as follows:

$$C_k^{NCP(n)} - \lambda = 0. \quad (14)$$

Replacing $C_k^{NCP(n)}$ with $C_k^{NCP(n)} - \lambda$ in Eq. (13) yields the following equation:

$$\frac{d\psi}{dt} - (C_k^{NCP(n)} - \lambda) = 0. \quad (15)$$

Herein, to address the material volume constraint, the Lagrange multiplier λ is computed employing a bisection algorithm. Theoretically, the LSF update reaches convergence once all nodal compliance proportion values equal λ . This equilibrium point, signified by the OC term $(C_k^{NCP(n)} - \lambda)$ equalling 0, constitutes a KKT point. Consequently, the LSF evolution in this scheme can also be mathematically modelled as follows:

$$\psi_{n+1} = \psi_n + \Delta t (C_k^{NCP(n)} - \lambda), \quad (16)$$

where Δt denotes the designer-specified time step. The algorithm terminates when either the compliance variation across consecutive iterations drops below a present threshold or the predefined maximum iteration number (IN_{max}) is reached:

$$\frac{|C_n - C_{n-r}|}{C_n} \leq \tau, \text{ or } n > IN_{max}, \quad (17)$$

in which τ represents a small positive tolerance value (typically 0.001), and r denotes an integer that is assigned a value of 5 in this study. This indicates that the compliance value exhibits negligible change over the final 5 iterations.

3. Reliability-Based Proportional Topology Optimisation Design

In practical engineering, excessive stress is a major trigger for structural deterioration and unexpected accidents. It should be clarified that the present work concentrates on compliance-minimised reliability topology optimisation, and explicit stress constraints are not incorporated into the current RPTO formulation. Even so, the probabilistic reliability constraint implemented in this work evaluates structural failure probability subjected to multi-source uncertainties including geometry, material and load fluctuations, which can indirectly reduce the risk of structural failure induced by adverse mechanical responses. Stress-constrained RPTO will be investigated as one of our important future research directions.

3.1. Variable description in RBTO problem

In this study, uncertainties are initially modelled using normal distributions to establish the computational framework. However, the proposed RPTO method utilises the Rosenblatt inverse transformation. A significant advantage of this transformation is its mathematical flexibility; it allows the framework to map non-normal distributions – such as the Beta distribution often associated with bounded manufacturing tolerances – into a standard normal space for reliability index calculation. This ensures that the RPTO method remains generalizable to diverse engineering statistical models. Within the context of this work, the structural RBTO problem involves two types of variables, specifically deterministic variables and uncertain variables. For deterministic variables, in the TO design of structural reliability using SIMP as the material interpolation model, density var-

iables applied to discrete elements are typically used to describe the configurational features of the optimised structure. For uncertain variables, the methods to describe their uncertainty can be categorised as probabilistic or non-probabilistic approaches. Since this paper studies RBTO of continuum structures under uncertainties in structural geometric dimensions, material volume, and applied loads, the structure's length, width, material volume fraction, and applied loads are considered uncertain variables. Their uncertainties are characterised using probabilistic methods. When these variables do not follow a standard normal distribution, they can be transformed into standard normal random variables through regularisation.

3.2. Mathematical model for RBTO

Based on the mathematical model in Eq. (1) for TO and referencing published work [24], we establish an RBTO model. This model considers volume constraints and reliability constraints while minimising structural compliance C , which is given by:

$$\begin{cases} \min : C = \mathbf{U}^T \mathbf{K} \mathbf{U} \\ \text{s.t. : } \begin{cases} P_f \leq P_f^* \\ \mathbf{K} \mathbf{U} = \mathbf{F} \\ \sum_{m=1}^N \rho_m v_m - V_f = 0 \\ 0 \leq \rho_m \leq 1, (m = 1, 2, \dots, N) \end{cases} \end{cases}, \quad (18)$$

where P_f and P_f^* denote the failure probability and the allowable one of the structures, respectively. The most direct approach to obtaining the failure probability involves integrating over the joint probability density function of the random variables. However, this method not only increases computational difficulty but is also often impractical in most scenarios. Consequently, scholars have proposed methods for approximating the failure probability. Widely applied techniques include Monte Carlo simulation, the First-Order Reliability Method (FORM), and the Second-Order Reliability Method. Due to its computational efficiency, simplicity, effectiveness, and ease of implementation, FORM is adopted here to approximate the failure probability. The function for approximating P_f and P_f^* can be formulated as:

$$\begin{cases} P_f \approx \Phi(-\beta) \\ P_f^* \approx \Phi(-\beta_i) \\ P_f \leq P_f^* \Leftrightarrow \beta \geq \beta_i \end{cases}, \quad (19)$$

where β and β_i denote the structural reliability index and the target reliability index, respectively. And $\Phi(\cdot)$ represents the cumulative distribution function of the standard normal distribution.

Furthermore, Eq. (18) can be rewritten as:

$$\begin{cases} \min : C = \mathbf{U}^T \mathbf{K} \mathbf{U} \\ \text{s.t. : } \begin{cases} \beta(\mathbf{u}) \geq \beta_t \\ \mathbf{K} \mathbf{U} = \mathbf{F} \\ \sum_{m=1}^N \rho_m v_m - V_f = 0 \\ 0 \leq \rho_m \leq 1, (m = 1, 2, \dots, N) \end{cases} \end{cases}, \quad (20)$$

in which $\mathbf{u}$ is a vector of random variables that follow a normal distribution.

3.3. Theoretical mapping of uncertainties

The transition from reliability analysis to deterministic optimisation is governed by the identification of the Most Probable Point (MPP) of failure. In this framework, the corrected variables x_i^* do not represent the "optimal" dimensions of the structure, but rather the worst-case scenario

realisations of the random variables that the structure must be able to withstand to meet the target reliability index β .

For a beam-like structure, an increase in the length L or a decrease in the material volume V_f typically results in increased compliance (reduced stiffness). To ensure a reliability level β , the reliability analysis stage shifts the mean values u_{xi} to the "fail-safe" bounds x_i^* . Consequently:

Geometric Parameters: The corrected length L^* increases because the structure must remain safe even if manufacturing tolerances result in a longer, more flexible span.

Volume Fraction: The corrected V_f^* decreases to represent the "worst-case" scenario of a material-deficient structure. By performing the equivalent deterministic topology optimisation (EDTO) on these penalised parameters, the final topology is inherently robust against the identified probabilistic variations.

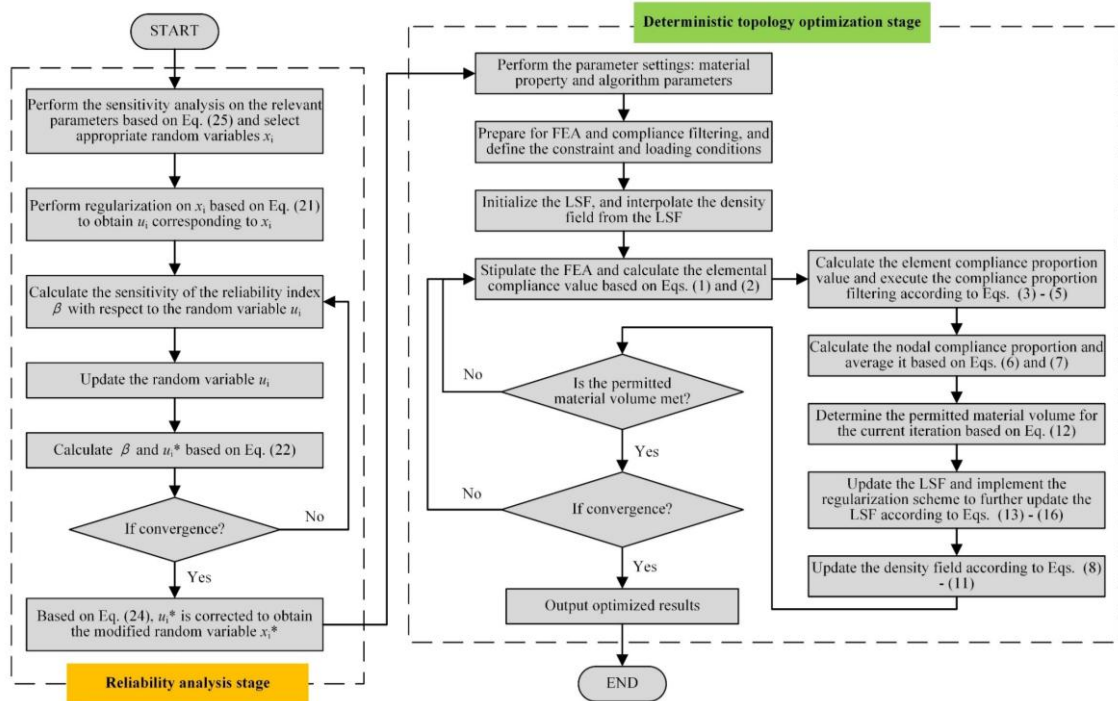


Fig. 1 Flowchart of the RPTO approach

3.4. Solution process for the RBTO model

This study utilises a hybrid decoupling approach to separate the RBTO problem into two distinct subproblems: reliability analysis and EDTO. During the reliability analysis phase, following the geometric meaning of β , the i -th random variable u_i^* that satisfies the reliability constraint and conforms to a normal distribution is obtained. Subsequently, u_i^* is corrected using the Rosenblatt inverse transformation [29], yielding the i -th random variable x_i^* that meets the reliability constraint and applies to the EDTO. In the EDTO phase, with x_i^* as the design parameter, the optimisation problem is addressed by means of the PTO method integrated with an LSF, resulting in an optimised structure that meets the predefined design requirements. Fig. 1 displays the flowchart of the RPTO approach. As a note, the convergence criterion for the reliability analysis phase is defined as $0 < \beta - \beta_t < 0.01$, while that for the EDTO phase is

given by Eq. (17).

To construct an optimisation model that maps random variable x_i to a normally distributed variable u_i , normalisation must first be applied to x_i using the following mathematical formulation:

$$u_i = \frac{x_i - \mu_{xi}}{\sigma_{xi}}, \quad (21)$$

where μ_{xi} and σ_{xi} denote the mean and standard deviation of the i -th random variable x_i , respectively.

Further, an optimisation model for u_i can be constructed based on Eq. (21):

$$\begin{cases} \beta = \min \|\mathbf{u}\| = \sqrt{\sum_{i=1}^I u_i^2} \\ \text{s.t. : } \beta(\mathbf{u}) \geq \beta_t \end{cases} \quad (22)$$

in which $u_i \in \mathbf{u}$, $i = 1, 2, \dots, I$.

When solving Eq. (22), the sensitivity of the reliability index β with respect to the random variable u_i is required, which can be obtained as follows:

$$\frac{\partial \beta}{\partial u_i} = \frac{1}{2 \left(\sqrt{\sum_{i=1}^I u_i^2} \right)} (2u_i) = \frac{u_i}{\beta}. \quad (23)$$

Solving Eq. (22) yields the random variable u_i^* that satisfies the reliability constraints and follows a normal distribution, which is known as the normalised value. Subsequently, the Rosenblatt inverse transformation is applied for correction, and its expression is as follows:

$$\begin{cases} x_i^* = \mu_{x_i} + u_i^* \sigma_{x_i}, & \text{for } \frac{\partial c}{\partial \mu_{x_i}} \geq 0 \\ x_i^* = \mu_{x_i} - u_i^* \sigma_{x_i}, & \text{for } \frac{\partial c}{\partial \mu_{x_i}} < 0 \end{cases}, \quad (24)$$

where x_i^* denotes the i -th random variable that satisfies the reliability constraints, namely the corrected value.

The sensitivity of the objective function (structural compliance c) with respect to the mean value μ_{x_i} of x_i can be approximated by the finite difference method:

$$\frac{\partial c}{\partial \mu_{x_i}} = \frac{c(\mu_{x_i} + \Delta \mu_{x_i}) - c(\mu_{x_i})}{\Delta \mu_{x_i}}, \quad (25)$$

in which $\Delta \mu_{x_i} = 0.01 \mu_{x_i}$.

4. Numerical Examples

The selected numerical benchmarks represent typical beam-like structural components widely encountered in engineering practice. The half-MBB beam corresponds to simply-supported load-bearing beams such as machine frame cross-members, while the cantilever beam mimics cantilever brackets and overhanging support arms. All these test cases consider concentrated external load conditions, which are representative loading scenarios for many practical mechanical components.

Several numerical examples including 2D cantilever and half-MBB beams and 3D half-MBB beam for tackling the minimum compliance case are presented for verifying the RPTO method's effectiveness. Herein, the length L , width W , thickness H , external load F and material volume fraction V_f of the structure are all set as mutually independent normal random variables. As a note, the uncertainties of the structural length, width and thickness are characterised by the uncertainty of the number of discrete elements in the x , y and z axial directions after the discretisation of the structural design domain. The main design parameter settings for the RPTO method in this paper are as follows: Young's modulus of the solid material $K_s = 1.0$ GPa, the external load $F = 100$ N, Poisson's ratio $\nu = 0.3$, and the filter radius r_f is set to 3 and 1.5 in the 2D and 3D examples, respectively. The other detailed parameters can be found in the published paper [22].

4.1. 2D examples

For both component types considered here, the highest stresses in service occur near the supports and beneath the point of load application. Explicit stress constraints lie outside the scope of the present formulation, as stated in Section 3, but these are the regions that govern the practical acceptability of the layouts obtained below.

The 2D models of half-MBB and cantilever beams are displayed in Fig. 2. The initial values of L and W for the half-MBB beam are assigned 90 and 40, respectively. The initial values of L and W for the cantilever beam are assigned 90 and 60, respectively. The initial values of material volume fraction V_f for the half-MBB and cantilever beams are endowed with 0.4 and 0.35, respectively. Assuming that the number of discrete elements L (i.e. x_1), W (i.e. x_2), external load F (i.e. x_3), and material volume fraction V_f (i.e. x_4) are all random variables. For each random variable x_i ($i = 1, 2, 3, 4$), the standard deviation σ_{x_i} is related to its mean μ_{x_i} by $\sigma_{x_i} = 0.1 \mu_{x_i}$.

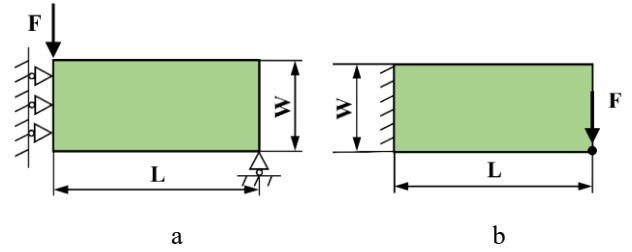


Fig. 2 2D example models: a – 2D half-MBB beam, b – 2D cantilever beam

During the reliability analysis phase, when the reliability index β takes values of 1, 2, 3, and 4, the random variables u_i^* (i.e., normalised values) satisfying the reliability constraints and following a normal distribution were obtained by solving the optimisation model Eq. (22). These were then processed via inverse transformation correction to derive the corrected values x_i^* for the random variables L , W , F , and V_f . The specific results are presented in Table 1 (2D half-MBB beam) and Table 2 (2D cantilever beam). As can be seen from Tables 1 and 2 there are differences between the corrected values and initial values of the random variables L , W , F , and V_f . Varying values of the structural reliability index β lead to different corrected values for these random variables. Furthermore, the larger the value of β , the greater the difference between the corrected values and initial values of the random variables.

The corrected values x_i^* for the random variables L , W , F , and V_f are incorporated into the EDTO stage as design parameters. TO is then performed on the 2D MBB and cantilever beams. The resulting structural topologies of the MBB and cantilever beams under different conditions are shown in Figs. 3 and 4, respectively. Figs. 3, a and 4, a show the topological structures of the MBB and cantilever beams obtained via deterministic TO (without considering reliability), respectively. Correspondingly, Figs. 3, b-e and 4, b-e show the topological structures of these two beam types obtained with different reliability index values considered. As can be observed from Figs. 3 and 4, the structures obtained via RBTO feature more supporting arm structures than those from deterministic TO. From an engineering perspective, MBB-type load-bearing beams are commonly implemented

Table 1
Corrected values of random variables obtained by
optimising 2D half-MBB beam

β	Random variables	Initial values	Regularised values	Corrected values
1	$L(x_1)$	90	0.556	95
	$W(x_2)$	40	0.500	38
	$F(x_3)$	100	0.476	104.76
	$V_f(x_4)$	0.400	0.476	0.381
2	$L(x_1)$	90	1.000	99
	$W(x_2)$	40	1.000	36
	$F(x_3)$	100	1.007	110.07
	$V_f(x_4)$	0.400	1.007	0.360
3	$L(x_1)$	90	1.444	103
	$W(x_2)$	40	1.500	34
	$F(x_3)$	100	1.531	115.31
	$V_f(x_4)$	0.400	1.531	0.339
4	$L(x_1)$	90	2.000	108
	$W(x_2)$	40	2.000	32
	$F(x_3)$	100	2.003	120.03
	$V_f(x_4)$	0.400	2.003	0.320

Table 2
Corrected values of random variables obtained by
optimising 2D cantilever beam

β	Random variables	Initial values	Regularised values	Corrected values
1	$L(x_1)$	90	0.556	95
	$W(x_2)$	60	0.500	57
	$F(x_3)$	100	0.476	104.76
	$V_f(x_4)$	0.350	0.476	0.333
2	$L(x_1)$	90	1.000	99
	$W(x_2)$	60	1.000	54
	$F(x_3)$	100	1.007	110.07
	$V_f(x_4)$	0.350	1.007	0.315
3	$L(x_1)$	90	1.444	103
	$W(x_2)$	60	1.500	51
	$F(x_3)$	100	1.531	115.31
	$V_f(x_4)$	0.350	1.531	0.296
4	$L(x_1)$	90	2.000	108
	$W(x_2)$	60	2.000	48
	$F(x_3)$	100	2.003	120.03
	$V_f(x_4)$	0.350	2.003	0.280

as truss-like or rib-reinforced frameworks, while cantilever support components typically adopt stiffened bracket or triangular truss configurations. The additional supporting arms emerging in the RBTO layouts are consistent with these widely adopted engineering structural forms, thereby better aligning the optimised designs with practical manufacturing and application requirements.

4.2. 3D examples

Both two-dimensional and three-dimensional numerical examples are implemented in this work for hierarchical validation of the proposed RPTO framework. Two-dimensional topology-optimisation models adopt plane-stress assumptions, offering low computational overhead and convenient

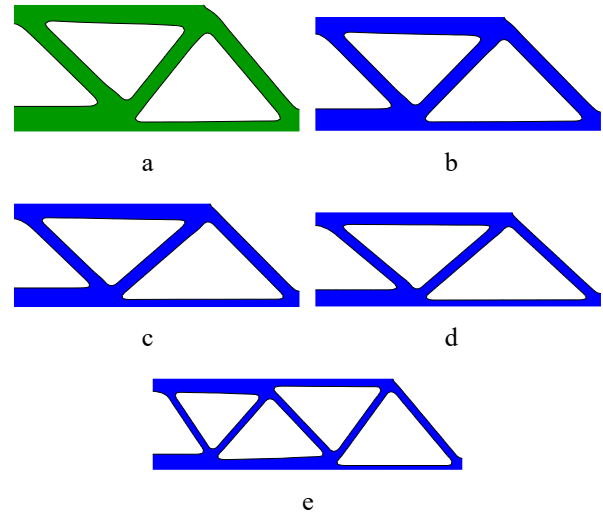


Fig. 3 Topology structures obtained by optimising a 2D half-MBB beam under different conditions: a – topological configuration from deterministic topology optimisation, b – topological configuration from RBTO ($\beta=1$), c – topological configuration from RBTO ($\beta=2$), d – topological configuration from RBTO ($\beta=3$), e – topological configuration from RBTO ($\beta=4$)

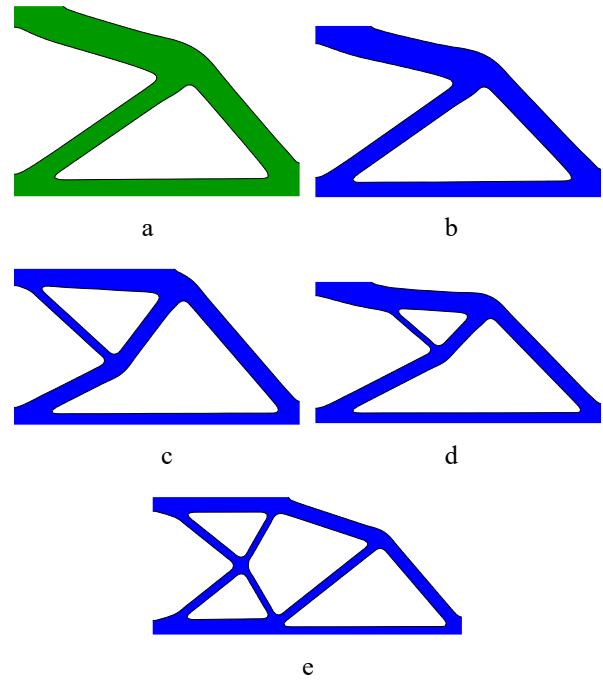


Fig. 4 Topology structures obtained by optimising a 2D cantilever beam under different conditions: a – topological configuration from deterministic topology optimisation, b – topological configuration from RBTO ($\beta=1$), c – topological configuration from RBTO ($\beta=2$), d – topological configuration from RBTO ($\beta=3$), e – topological configuration from RBTO ($\beta=4$)

observation of fundamental topological evolutionary behaviours under varying reliability-index levels. Nevertheless, 2D formulations cannot capture out-of-plane mechanical responses and thickness-related random-variable effects. By

contrast, 3D models enable full-volume mechanical representation of practical beam-like components and support random-variable descriptions for out-of-plane geometric dimensions, though they bring substantially higher computational costs due to increased element counts. In general, 2D models are suitable for conceptual investigation and parametric-sensitivity analysis of topology-optimisation algorithms, while 3D models are preferred when out-of-plane loading, thickness-dependent uncertainties or practical component-oriented evaluation are required. In the present study, 2D benchmarks are first utilised to explore the intrinsic mechanism of the reliability-based topology-optimisation approach, and the 3D half-MBB beam example further demonstrates the practicability of the developed method for realistic three-dimensional structural design.

Fig. 5 shows the 3D model of a half-MBB beam. The initial values of L , W , and H for the 3D MBB beam are set to 40, 20, and 20, respectively. The initial value of material volume fraction V_f for the 3D MBB beam is assigned 0.25. Assuming that the number of discrete elements L (i.e. x_1), W (i.e. x_2), H (i.e. x_3), external load F (i.e. x_4), and material volume fraction V_f (i.e. x_5) are all random variables. For each random variable x_i ($i = 1, 2, 3, 4, 5$), the standard deviation σ_{xi} is related to its mean μ_{xi} by $\sigma_{xi} = 0.1 \mu_{xi}$.

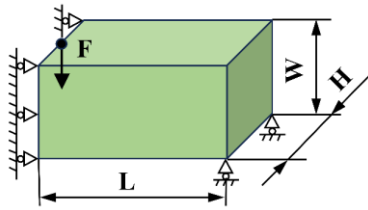


Fig. 5 3D model for half-MBB beam

During the reliability analysis stage for the structural TO of the 3D MBB beam, when the reliability index β takes values of 1, 2, 3, and 4, the random variables u_i^* (i.e., normalised values) satisfying the reliability constraints and following a normal distribution were obtained by solving the optimisation model Eq. (22). These were then processed via inverse transformation correction to derive the corrected values x_i^* for the random variables L , W , H , F , and V_f . The specific results are presented in Table 3. As is evident from Table 3, there are differences between the corrected values and initial values of the random variables L , W , H , F , and V_f . Varied values of the structural reliability index β induce different corrected values for these random variables. Besides, as the value of β increases, the greater the difference between the corrected values and the initial values of the random variables grows.

Similar to the calculation process of the 2D numerical example, the corrected values x_i^* for the random variables L , W , H , F , and V_f are incorporated into the EDTO stage as design parameters. TO is subsequently carried out on the 3D half-MBB beam. And the resulting structural topologies of the 3D MBB beam under different conditions are displayed in Fig. 6. Fig. 6, a presents the topological structure of the 3D MBB beam acquired through deterministic TO (with no consideration of reliability). Accordingly, Figs. 6, b-e present the topological structures of the 3D MBB beam acquired by considering different reliability index values. As can be clearly seen from Fig. 6, the structure obtained

Table 3
Corrected values of random variables obtained by optimising 3D half-MBB beam

β	Random variables	Initial values	Regularised values	Corrected values
1	$L(x_1)$	40	0.500	42
	$W(x_2)$	20	0.500	19
	$H(x_3)$	20	0.500	19
	$F(x_4)$	100	0.366	103.66
	$V_f(x_5)$	0.250	0.366	0.241
2	$L(x_1)$	40	1.000	44
	$W(x_2)$	20	1.000	18
	$H(x_3)$	20	1.000	18
	$F(x_4)$	100	0.714	107.14
	$V_f(x_5)$	0.250	0.714	0.232
3	$L(x_1)$	40	1.250	45
	$W(x_2)$	20	1.500	17
	$H(x_3)$	20	1.500	17
	$F(x_4)$	100	1.215	112.15
	$V_f(x_5)$	0.250	1.215	0.220
4	$L(x_1)$	40	1.750	47
	$W(x_2)$	20	2.000	16
	$H(x_3)$	20	2.000	16
	$F(x_4)$	100	1.581	115.81
	$V_f(x_5)$	0.250	1.581	0.211

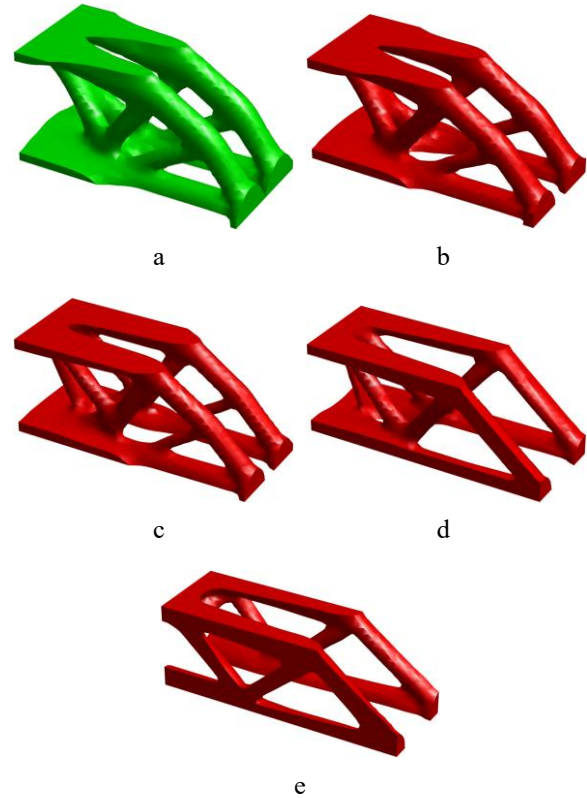


Fig. 6 Topology structures obtained by optimising 3D half-MBB beam under different conditions: a – topological configuration from deterministic topology optimisation, b – topological configuration from RBTO ($\beta = 1$), c – topological configuration from RBTO ($\beta = 2$), d – topological configuration from RBTO ($\beta = 3$), e – topological configuration from RBTO ($\beta = 4$)

when the reliability index β equals 1 is similar to that derived from deterministic TO. When the reliability index β is in the range of 2–4, the obtained structure becomes simpler as the amount of material reduces (with a larger β corresponding to less material). Particularly when β takes a value of 4, the obtained optimised structures not only exhibit a more regular morphology but also are more compatible with the requirements of practical engineering applications. In addition, it can also be observed from the data in Table 3 that, when reliability is considered (i.e., when the β values are 1, 2, 3, and 4), the material volume fraction of the obtained optimised structure decreases by 3.6%, 7.2%, 12%, and 15.6% respectively, compared with the case when reliability is not considered (where $V_f = 0.25$).

4.3. Reliability validation via Monte Carlo simulation

To verify the accuracy of the First-Order Reliability Method (FORM) used in the RPTO process, a post-optimisation validation was conducted. Taking the optimised 2D half-MBB beam ($\beta = 3$) as a representative case, a Monte Carlo (MC) simulation was performed with 10^6 samples based on the initial mean and standard deviation values of the random variables. The calculated probability of failure ($P_{f, MCS} = 0.001346$) was found to be in close agreement with the target failure probability ($P_i^* = \Phi(-3) = 0.001330$). Fig. 7 presents the histogram of limit state function value distribution based on MC sampling, which accurately verifies the effectiveness of the FORM from the perspective of statistical distribution. This figure includes two subgraphs of the original spherical $g_{sph}(U)$ and the tangent plane $g_{cut}(U)$. The red dashed line represents the failure threshold $g(U) = 0$, where a sample is judged as a failure if $g(U) < 0$. In the left subgraph, the distribution of $g_{sph}(U)$ derived from MC sampling exhibits an overall rightward shift with a positive mean value, indicating that the mean point (i.e., the origin) of the standardised random variables lies within the reliable region. This is fully consistent with the basic assumption of FORM, confirming the accuracy of MC as a benchmark method for reliability analysis. In the right subgraph, $g_{cut}(U)$ follows a normal distribution with a mean of 3.0045 and a standard deviation of 1.0, with its peak centred around 3.0045. The failure threshold $g(U) = 0$ resides in the left tail of the distribution, corresponding to a failure sample proportion of about 0.135%. This proportion is completely consistent with the FORM theoretical failure probability $\Phi(-\beta)$,

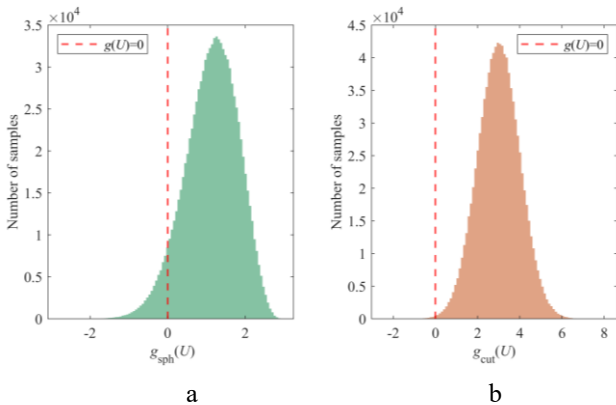


Fig. 7 Histogram of limit state function value distribution based on MC sampling: a – original spherical $g_{sph}(U)$, b – tangent plane $g_{cut}(U)$

which directly validates the high accuracy of FORM in approximating the original limit state surface using the tangent plane. In conclusion, this confirms that the decoupled RPTO approach accurately captures the intended reliability level while maintaining computational efficiency significantly superior to direct simulation-based optimisation methods.

5. Conclusions

In this work, we proposed a new method, referred to as RPTO, to tackle the RBTO problem of continuum structures with material volume and reliability constraints for minimum structural compliance. Several 2D and 3D cases were employed to demonstrate the effectiveness of the newly proposed RPTO method. From the numerical examples presented in this paper, we may arrive at the following conclusions:

The proposed RPTO approach can effectively solve the RBTO problem for continuum structures, and the resulting optimised structures exhibit smooth boundaries.

Under conditions where uncertainties exist in structural geometric dimensions, material volume, and applied loads, the optimised structures obtained via the RPTO method are not only more regular in morphology but also more compatible with the requirements of practical engineering applications, compared with those derived from deterministic TO design.

The selection of the reliability index β in structural RBTO directly affects the corrected values of random variables (including geometric dimensions, material volume, and applied loads), thereby influencing the resulting topological configuration and material distribution. Moreover, greater β values progressively result in more rational material layouts. The proposed RPTO framework is promising for the reliability-oriented conceptual design of beam-type load-bearing components in mechanical equipment, vehicle frames and general lightweight structures where multi-source geometric, material and load uncertainties cannot be neglected.

Acknowledgments

This work was supported by the National Natural Science Foundation of China (Grant No. 51675450), and the Sichuan Science and Technology Program (No. 2023YFG0182).

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L. Yang, Y. Song, X. Hu, X. Rao, R. Du, W. Cheng

RELIABILITY-BASED PROPORTIONAL TOPOLOGY OPTIMISATION OF CONTINUUM STRUCTURES WITH SMOOTH BOUNDARIES

S u m m a r y

This study presents a reliability-based proportional topology optimisation (RPTO) framework for continuum

structures, specifically designed to achieve smooth, manufacturing-ready boundaries under multi-source uncertainties. By integrating the Proportional Topology Optimisation (PTO) algorithm with a Level Set Method (LSM), the approach bypasses the need for high-order sensitivity information while ensuring high-fidelity boundary representation. A hybrid decoupling strategy is employed to decompose the reliability-based topology optimisation (RBTO) problem into a reliability analysis sub-problem and an equivalent deterministic topology optimisation (EDTO) sub-problem. Uncertainties in geometry, material volume, and loading are characterised via a probabilistic framework using the First-Order Reliability Method (FORM). Numerical results for 2D and 3D benchmarks demonstrate that the RPTO method effectively captures the influence of the target reliability index on structural morphology, yielding robust designs that satisfy strict probabilistic constraints without the "grey areas" common in density-based methods.

Keywords: reliability-based proportional topology optimisation, first-order reliability method, level-set function, compliance minimisation.

Received January 28, 2026

Accepted August 17, 2026



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