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On the behaviour of current-carrying wire-conductors and bucking of a column

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1. Introduction

In the last few decades, Single-Degree-Of-Freedom (SDOF) oscillator has been widely used to study the behavior of machines used in pile driving, compacting, rock drilling, impact printing and marine structures [1-4].

Dario Aristizabal-Ochoa analyzed the large-deformation-small strain and postbuckling behavior of Timoshenko beam-columns subjected to conservative as well as non-conservative end loads. He investigated the combined effects of shear, axial and bending deformations in a simplified manner. Later, a one-dimensional composite frame element for nonlinear static and cyclic behavior of concrete-filled steel beam columns is formulated by Valipour and Foster [5].A nonlinear fiber element analysis is presented through the work presented by Liang et al. [6] for predicting the ultimate strengths of thin-walled steel box columns with local buckling behavior.

Apart from the studies mentioned above, many works have been carried out to analyze the nonlinear vibrations, most of which on developing governing equations for Dynamic response of axially loaded Euler-Bernoulli beams [7, 8], inextensible beams [9, 10], transportation [11], cubic-quintic Duffing [12, 13], mass-spring systems [14], and more [15-23].

In addition, the ability to determine the magnetic fields and the resulting parameters (force, impedance, power losses) is very important in the optimization of electric machines and equipment. Gasiorski in 1986 [24] presented a general method which is based on combination of Bubnov-Galerkin methods by means of finite element method. The presented approach was utilized for calculating impedance of polygonal and symmetrical shape conductors carrying current chosen for simulations.

In the present paper, we obtain an approximate expression for the periodic solutions to two practical cases [25, 26] of nonlinear SODF oscillation systems, namely oscillation of current-carrying wire in a magnetic field and the model of bucking of a column by means of iteration perturbation method (IPM), variational approach (VA), and perturbation expansion method (PEM). These techniques yield a very rapid convergence using an iteration and lead to high accuracy of the solution. The results presented in this paper reveal that these methods are very effective and convenient for conservative nonlinear oscillators.

2. The models of nonlinear SODF systems

2.1. Case 1: Motion of a current-carrying conductor

Fig. 1 shows a pair of current-carrying wire-conductors restrained by a wire to a fixed wall by linear elastic springs. Assume $\tilde{x}$, k, and m as displacement of the wire, stiffness of the springs and mass of wire respectively. The differential equation describing the motion of wire is [25]:

$$m\frac{d^2\tilde{x}^2}{dt^2}+k\tilde{x}-\frac{2i_1i_2l}{b-\tilde{x}}=0,\quad x(0)=\tilde{A},\quad \frac{dx}{dt}(0)=0\,, \qquad (1)$$

where $k\tilde{x}$ the restoring forces due to is springs and $2i_1i_2l/(b-\tilde{x})$ is the attraction force between the conductors due to magnetic fields produced by the currents. Eq. (1) can be rewritten as a conservative nonlinear oscillatory system with a rational form:

$$\frac{d^2x}{dt^2}+x-\frac{\Delta}{1-x}=0,\quad x(0)=\tilde{A},\quad \frac{dx}{dt}(0)=0\,, \qquad (2)$$

where $x=\tilde{x}/b$, $t=\omega_0\tilde{t}$, $\omega_0^2=k/m$, and $\Delta=2i_1i_2l/kb^2$. $\tilde{A}$ is also the initial condition for x.

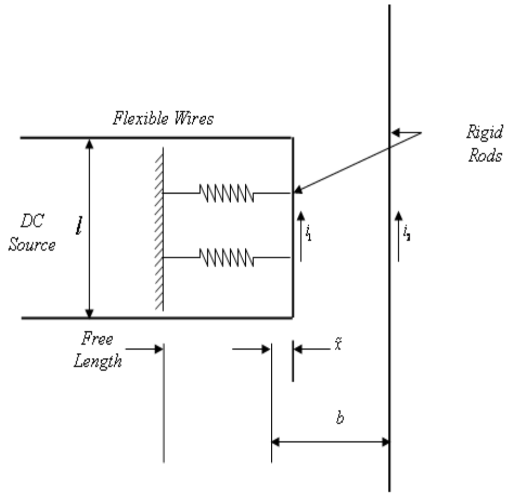

Fig. 1 Current-carrying wire in the field of an infinite current-carrying conductor [25]

The following four cases should be separately considered: $\Delta < 0$, $\Delta = 0$, $0 < \Delta < 1/4$ and $\Delta \geq 1/4$ [25]. Authors are interested in constructing analytical approximate periodic solutions to Eq. (2). Three different approximate as IPM, VA, and PEM are utilized to construct analytical approximations to periodic oscillation of the current-carrying wire for the cases $\Delta < 0$ and $0 < \Delta < 1/4$.

The first-order approximate procedure yields rapid convergence with respect to the "exact" solution obtained by numerical integration. In addition, the results are valid for all permitted oscillation amplitude.

2.2. Case 2: Model of a buckling column

In this section we consider the structure exposed to buckling as shown in Fig. 2. The mass *m* moves in the horizontal direction only. It is therefore studied the static stability by determining the nature of the singular point at $x = 0$ of the dynamic equations. The proposed dynamic approach is more convenient and effective to use than the static concept [26].

Neglecting the weight of all but the mass, show that the governing equation for the motion of *m* is [26]:

$$m\ddot{u} + \left(k_1 - \frac{2P}{l}\right)u + \left(k_3 - \frac{2P}{l^3}\right)u^3 + ... = 0\,. \tag{3}$$

Where the spring force is given by:

$$F_{spring} = k_1 u + k_3 u^3 + ... \tag{4}$$

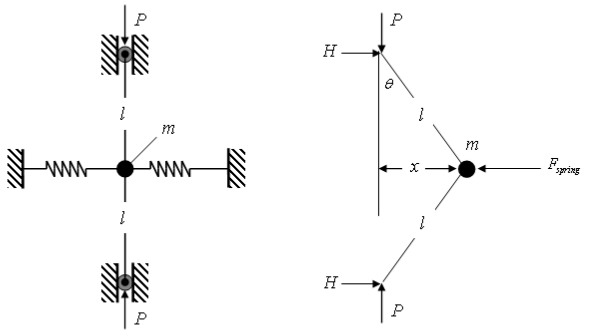

Fig. 2 Model for the bulking of a column [26]

3. Solution procedures

3.1. Basic idea of IPM

In this paper, we will consider the second-order differential equation:

$$\ddot{u} + f(u,t) = 0\,. \tag{5}$$

We introduce the variable $y = du/dt$, and then Eq. (5) can be replaced by equivalent system:

$$\dot{u}(t) = y(t); \tag{6}$$

$$\dot{y}(t) = -f(u,t)\,. \tag{7}$$

Assume that its initial approximate guess can be expressed as:

$$u(t) = A\cos(\omega t), \tag{8}$$

where *w* is the angular frequency of the oscillation. Then we have:

$$\dot{u}(t) = -A\omega \sin(\omega t) = y(t)\,. \tag{9}$$

Substituting Eq. (8) and 9 into the Eq. (7), we obtain:

$$\dot{y}(t) = -f\left(A\cos(\omega t), t\right). \tag{10}$$

Using Fourier expansion series in the right hand of Eq. (10):

$$f\left(A\cos(\omega t), t\right) = \sum_{n=0}^{\infty} a_{2n+1}\cos\left[(2n+1)\omega t\right] = \\ = \alpha_1 \cos(\omega t) + \alpha_3 \cos(3\omega t) + ... \tag{11}$$

Substituting Eq. (11) into Eq. (10) yields:

$$\dot{y}(t) = -\left(\alpha_1 \cos(\omega t) + \alpha_3 \cos(3\omega t) + ...\right). \tag{12}$$

Integrating Eq. (12), yields:

$$y(t) = -\frac{\alpha_1}{\omega}\sin(\omega t) - \frac{\alpha_3}{3\omega}\sin(3\omega t) - ... \tag{13}$$

Comparing Eq. (9) and (13), we obtain:

$$-A\omega = -\frac{\alpha_1}{\omega}; \tag{14}$$

$$\omega = \sqrt{\frac{\alpha_1}{A}}; \tag{15}$$

$$T = 2\pi\sqrt{\frac{A}{\alpha_1}}\,. \tag{16}$$

3.2. Basic idea of VA

For explaining the VA procedure, we consider a general nonlinear oscillator in the form of Eq. (5). Its variational principle can be established using the semi-inverse method [27, 28]:

$$J(u) = \int_0^{T/4}\left(-\frac{1}{2}\dot{u}^2 + F(u)\right)dt, \tag{17}$$

where T is period of the nonlinear oscillator, $F(u) = \int f(u)\,du$. Assume that its solution can be expressed as Eq. (8). Substituting (8) into (17) results in:

$$J(A,\omega) = \int_0^{T/4}\left(-\frac{1}{2}A^2\omega^2 \sin^2 \omega t + F\left(A\cos\omega t\right)\right)dt = \\ = \frac{1}{\omega}\int_0^{\pi/2}\left(-\frac{1}{2}A^2\omega^2 \sin^2 t + F\left(A\cos t\right)\right)dt = \\ = -\frac{1}{2}A^2\omega\int_0^{\pi/2}\sin^2 t\,dt + \frac{1}{\omega}\int_0^{\pi/2}F\left(A\cos\omega t\right)dt\,. \tag{18}$$

Applying the Ritz method, we require:

$$\frac{\partial J}{\partial A}=0; \quad (19)$$

$$\frac{\partial J}{\partial \omega}=0. \quad (20)$$

But using a careful inspection, for most cases we find:

$$\frac{\partial J}{\partial \omega}=-\frac{1}{2}A^2\int_0^{\pi/2} sin^2 t\, dt-\frac{1}{\omega^2}\int_0^{\pi/2} F\left(A cos t\right)dt<0. \quad (21)$$

Thus, we modify conditions (19) and (20) into a simple form:

$$\frac{dJ}{d\omega}=0. \quad (22)$$

3.3. Basic idea of PEM

In order to use PEM, we rewrite the general form of Duffing equation in Eq. (5) in the following form [9]:

$$\ddot{u}+\alpha u+\beta N(u,t)=0, \quad (23)$$

where $N(u,t)$ is the nonlinear term after expending the solution u; α as a coefficient of u and β as a coefficient of $N(u,t)$, the series of p introduce as follows:

$$u=u_0+pu_1+p^2u_2+\ldots; \quad (24)$$

$$\alpha=\omega^2+p\gamma_1+p^2\gamma_2+\ldots; \quad (25)$$

$$\beta=p\delta_1+p^2\delta_2+\ldots \quad (26)$$

Substituting Eqs. (24) - (26) into Eq. (23) and equating terms with the identical powers of *p*, we have:

$$p^0: \quad \ddot{u}_0+\omega^2 u_0=0; \quad (27)$$

$$p^1: \quad \ddot{u}_1+\omega^2 u_1+\gamma_1 u_0+\delta_1 N\left(u_0,t\right)=0. \quad (28)$$

Considering initial conditions $u_0\left(0\right)=A$ and $\dot{u}_0\left(0\right)=0$, the solution of Eq. (27) is $u_0=A cos\left(\omega t\right)$. Substituting u_0 into Eq. (28), we obtain:

$$p^1: \quad \ddot{u}_1+\omega^2 u_1+\gamma_1 A cos\left(\omega t\right)+\delta_1 N\left(A cos\left(\omega t\right),t\right)=0. \quad (29)$$

Similar to IPM, for achieving the secular term, we use Fourier expansion series as follows:

$$\begin{aligned}&\delta_1 N\left(A cos\left(\omega t\right),t\right)=\\&=\sum_{n=0}^{\infty} b_{2n+1} cos\left[\left(2n+1\right)\omega t\right]=\\&=b_1 cos\left(\omega t\right)+b_3 cos\left(3\omega t\right)+\ldots\approx b_1 cos\left(\omega t\right)\end{aligned} \quad .(30)$$

Substituting Eq. (30) into Eq. (29) yields:

$$p^1: \quad \ddot{u}_1+\omega^2 u_1+\left(\gamma_1 A+b_1\right)cos\left(\omega t\right)=0. \quad (31)$$

For avoiding secular term, we have:

$$\left(\gamma_1 A+b_1\right)=0. \quad (32)$$

Setting $p=1$ in Eqs. (25) and (26), and Substituting $\gamma_1=\alpha-\omega^2$ and $\delta_1=\beta$ in Eq. (32), we can achieve the frequency and period of Eq. (5).

4. Applications of analytical solutions for Eq. (2)

To show the applicability, accuracy and effectiveness of proposed methods, they are applied to the first practical case presented in Eqs. (2). We use the simple form of Eq. (2) to obtain the approximate solutions based on IPM, VA, and PEM. For this sake, we let $x=x_2+u$ in Eq. (2) and expand the resulting equation in a Taylor series about $u=0$. The result is:

$$\ddot{u}+\alpha_1 u+\alpha_2 u^2+\alpha_3 u^3+\ldots=0, \; x\left(0\right)=\tilde{A}, \; \dot{x}\left(0\right)=0, \quad (33)$$

where

$$\alpha_1=1-\frac{\Delta}{\left(1-\alpha\right)^2}, \quad \alpha_2=-\frac{\Delta}{\left(1-\alpha\right)^3}, \quad \alpha_3=-\frac{\Delta}{\left(1-\alpha\right)^4}. \quad (34)$$

4.1. Implementation of IPM

As it can be seen in the basic idea of IPM, after introducing the variable $y=du/dt$, and substituting $u=A cos\left(\omega t\right)$ into the Eq. (33), we obtain:

$$\dot{y}=-\alpha_1 A cos\left(\omega t\right)-\alpha_2 A^2 cos^2\left(\omega t\right)-\alpha_3 A^3 cos^3\left(\omega t\right). \quad (35)$$

By using Fourier series expansion, we have:

$$\begin{aligned}\dot{y}&=\sum_{n=0}^{\infty}\alpha_{2n+1} cos\left[\left(2n+1\right)\omega t\right]=\left[-\alpha_1 A-\left(\frac{4}{\pi}\int_0^{\pi/2} cos\left(\varphi\right)\left(\alpha_2 A^2 cos^2\left(\varphi\right)+\alpha_3 A^3 cos^3\left(\varphi\right)\right)d\varphi\right)\right]cos\,\omega t+\ldots=\\&=\left(-\frac{A\left(12\alpha_1\pi+32\alpha_2 A+9\alpha_3 A^2\pi\right)}{12\pi}\right)cos\,\omega t+\ldots\end{aligned} \quad (36)$$

By integrating Eq. (36), and comparing with Eq. (9), we obtain:

$$\omega = \sqrt{\left(\frac{\left(12\alpha_1\pi + 32\alpha_2 A + 9\alpha_3 A^2\pi\right)}{12\pi}\right)}; \tag{37}$$

$$T = \frac{4\sqrt{3}\pi^{3/2}}{\sqrt{\left(\left(12\alpha_1\pi + 32\alpha_2 A + 9\alpha_3 A^2\pi\right)\right)}}. \tag{38}$$

4.2. Implementation of VA

In this section, we will use the VA solution for Eq. (33). The variational principle of this equation can be obtained:

$$J(u) = \int_0^{T/4}\left(-\frac{1}{2}\dot{u}^2 + \int\left(\alpha_1 u + \alpha_2 u^2 + \alpha_3 u^3\right)du\right)dt. \tag{39}$$

Using a trial function $u = A\cos\omega t$ into (39), the solution of (33) can be expressed as follows:

$$J(A) = \int_0^{T/4}\begin{pmatrix} -\frac{1}{2}A^2\omega^2\sin^2\omega t + \frac{\alpha_1 A^2\cos^2\omega t}{2} + \\ +\frac{\alpha_2 A^3\cos^3\omega t}{3} + \frac{\alpha_3 A^4\cos^4\omega t}{4}\end{pmatrix}dt. \tag{40}$$

Thus, the stationary condition with respect to A is:

$$\frac{\partial J}{\partial A} = \int_0^{T/4}\begin{pmatrix} -A\omega^2\sin^2\omega t + \alpha_1 A\cos^2\omega t + \\ \alpha_2 A^2\cos^3\omega t + \alpha_3 A^3\cos^4\omega t\end{pmatrix}dt = 0. \tag{41}$$

This leads to the result:

$$\omega^2 = \frac{\int_0^{T/4}\begin{pmatrix}\alpha_1 A\cos\omega t + \alpha_2 A^2\cos^2\omega t + \\ +\alpha_3 A^3\cos^3\omega t\end{pmatrix}dt}{A\int_0^{T/4}\left(\sin^2\omega t\right)dt} = \frac{\int_0^{\pi/2}\begin{pmatrix}\alpha_1 A^2\cos^2 t + \alpha_2 A^3\cos^3 t + \\ +\alpha_3 A^4\cos^4 t\end{pmatrix}dt}{A\int_0^{T/4}\left(\sin^2 t\right)dt}. \tag{42}$$

Thus, we obtain the following frequency and period as same as IPM solution:

$$\omega = \sqrt{\left(\frac{\left(12\alpha_1\pi + 32\alpha_2 A + 9\alpha_3 A^2\pi\right)}{12\pi}\right)}; \tag{43}$$

$$T = \frac{4\sqrt{3}\pi^{3/2}}{\sqrt{\left(\left(12\alpha_1\pi + 32\alpha_2 A + 9\alpha_3 A^2\pi\right)\right)}}. \tag{44}$$

4.3. Implementation of PEM

In order to use PEM procedure, we can rewrite (33) in following form:

$$\ddot{u} + (\alpha_1)u + 1\left(\alpha_2 u^2 + \alpha_3 u^3\right) = 0. \tag{45}$$

This equation is same as the Eq. (23) form where $\alpha = \alpha_1$ and $\beta = 1$. According to PEM and Substituting $\alpha = \alpha_1$ and $\beta = 1$ into Eqs. (25)-(26), we have:

$$\alpha_1 = \omega^2 + p\gamma_1 + p^2\gamma_2 + \ldots; \tag{46}$$

$$1 = p\delta_1 + p^2\delta_2 + \ldots \tag{47}$$

Substituting Eqs. (24) and (46)-(47) into Eqs. (45) and equating the terms with the identical powers of p, we obtain:

$$p^0:\quad \ddot{u}_0 + \omega^2 u_0 = 0; \tag{48}$$

$$p^1:\quad \ddot{u}_1 + \omega^2 u_1 + \gamma_1 u_0 + \delta_1\alpha_2 u_0^2 + \delta_1\alpha_3 u_0^3 = 0. \tag{49}$$

Considering initial conditions $u(0) = A$ and $\dot{u}(0) = 0$, the solution of Eq. (48) is $u_0 = A\cos(\omega t)$. Substituting u_0 into Eq. (49), we obtain:

$$p^1:\quad \ddot{u}_1 + \omega^2 u_1 + \gamma_1 A\cos(\omega t) + \\ + \delta_1\alpha_2 A^2\cos^2(\omega t) + \delta_1\alpha_3 A^3\cos^3(\omega t) = 0. \tag{50}$$

It is possible to perform the following Fourier series expansion:

$$\delta_1\alpha_2 A^2\cos^2(\omega t) + \delta_1\alpha_3 A^3\cos^3(\omega t) = \sum_{n=0}^{\infty} a_{2n+1}\cos\left[(2n+1)\omega t\right] = \left(\alpha_1\cos(\omega t) + \ldots\right) = \\ = \left(\frac{4}{\pi}\int_0^{\pi/2}\cos(\varphi)\left(\delta_1\alpha_2 A^2\cos^2(\varphi) + \delta_1\alpha_3 A^3\cos^3(\varphi)\right)d\varphi\right)\cos(\omega t) + \ldots = \frac{A^2\delta_1\left(32\alpha_2 + 9A\alpha_3\pi\right)}{12\pi}. \tag{51}$$

Substituting Eq. (51) into Eq. (50) gives:

$$\ddot{u}_1 + \omega^2 u_1 + \left(\gamma_1 A + \frac{A^2\delta_1\left(32\alpha_2 + 9A\alpha_3\pi\right)}{12\pi}\right)\cos(\omega t) + \\ + \sum_{n=1}^{\infty} a_{2n+1}\cos\left[(2n+1)\omega t\right] = 0. \tag{52}$$

No secular term in u_1 requires that:

$$\gamma_1 A + \frac{A^2\delta_1\left(32\alpha_2 + 9A\alpha_3\pi\right)}{12\pi} = 0. \tag{53}$$

Setting $p = 1$ in Eqs. (25) and (26), we have:

$$\alpha_1 = \omega^2 + \gamma_1; \tag{54}$$

$$1 = \delta_1. \tag{55}$$

From Eqs. (53)-(55), we obtain:

$$\omega = \sqrt{\left(\frac{\left(12\alpha_1 \pi + 32\alpha_2 A + 9\alpha_3 A^2 \pi\right)}{12\pi}\right)}; \tag{56}$$

$$T = \frac{4\sqrt{3}\pi^{3/2}}{\sqrt{\left(\left(12\alpha_1 \pi + 32\alpha_2 A + 9\alpha_3 A^2 \pi\right)\right)}}. \tag{57}$$

5. Applications of analytical solutions for Eq. (3)

Similar to prior section, in this part, we applied the approximate methods for another practical case presented in Eq. (3). This equation can be put in the following general form:

$$\ddot{u} + \alpha_1 u + \alpha_3 u^3 = 0, \tag{58}$$

where $\alpha_1 = \left(\frac{k_1}{m} - \frac{2P}{l\,m}\right)$, $\alpha_3 = \left(\frac{k_3}{m} - \frac{2P}{l^3 m}\right)$.

5.1. Applying the IPM

After introducing the variable $y = du/dt$, and Substituting $u = A\cos(\omega t)$ into the Eq. (58), we obtain:

$$\dot{y} = -\alpha_1 A\cos(\omega t) - \alpha_3 A\cos^3(\omega t). \tag{59}$$

By using Fourier series expansion, we have:

$$\dot{y} = \sum_{n=0}^{\infty} \alpha_{2n+1} \cos\left[(2n+1)\omega t\right] =$$
$$= \left[-\alpha_1 A - \left(\frac{4}{\pi}\alpha_3 A^3 \int_0^{\pi/2} \left(\cos^4(\varphi)\right) d\varphi\right)\right]\cos(\omega t) + ... =$$
$$= \left(-\frac{A\left(12\alpha_1 \pi + 32\alpha_2 A + 9\alpha_3 A^2 \pi\right)}{12\pi}\right)\cos(\omega t) + ... \tag{60}$$

By integrating Eq. (60), and comparing with Eq. (9), we obtain:

$$\omega = \sqrt{\alpha_1 + \frac{3}{4}\alpha_3 A^2}; \tag{61}$$

$$T = \frac{4\pi}{\sqrt{4\alpha_1 + 3\alpha_3 A^2}}. \tag{62}$$

5.2. Applying the VA

In this section, we will use the Variational Approach solution for Eq. (58). The variational principle of Eq. (58), can be obtained:

$$J(u) = \int_0^{T/4} \left(-\frac{1}{2}\dot{u}^2 + \int\left(\alpha_1 u + \alpha_2 u^3\right) du\right) dt. \tag{63}$$

Using a trial function $u = A\cos\omega t$ into Eq. (63), the solution of Eq. (58) can be expressed as follows:

$$J(A) = \int_0^{T/4} \left(\begin{array}{l} -\frac{1}{2}A^2\omega^2 \sin^2 \omega t + \\ + \frac{\alpha_1 A^2 \cos^2 \omega t}{2} + \frac{\alpha_2 A^4 \cos^4 \omega t}{4} \end{array}\right) dt. \tag{64}$$

Thus, the stationary condition with respect to A is:

$$\frac{\partial J}{\partial A} = \int_0^{T/4} \left(\begin{array}{l} -A\omega^2 \sin^2 \omega t + \\ + \alpha_1 A\cos^2 \omega t + \alpha_2 A^3 \cos^4 \omega t \end{array}\right) dt = 0. \tag{65}$$

This leads to the result:

$$\omega^2 = \frac{\int_0^{T/4}\left(\alpha_1 A\cos^2 \omega t + \alpha_2 A^3 \cos^4 \omega t\right)dt}{A\int_0^{T/4}\left(\sin^2 \omega t\right)dt} =$$
$$= \frac{\int_0^{\pi/2}\left(\alpha_1 A\cos^2 t + \alpha_2 A^3 \cos^4 t\right)dt}{A\int_0^{T/4}\left(\sin^2 t\right)dt}. \tag{66}$$

Thus, we obtain the following frequency and period same as the IPM solution:

$$\omega = \sqrt{\alpha_1 + \frac{3}{4}\alpha_3 A^2}; \tag{67}$$

$$T = \frac{4\pi}{\sqrt{4\alpha_1 + 3\alpha_3 A^2}}. \tag{68}$$

5.3. Applying the PEM

We assume that in Eq. (58), $\alpha = \alpha_1$ and $\beta = \alpha_3$. Similar to sections 3.3 and 4.3, we expand the solution u, and its coefficients α_1 and 1 (Eqs. (24)-(26)), and substituting into Eq. (58), we can obtain:

$$p^0: \quad \ddot{u}_0 + \omega^2 u_0 = 0; \tag{69}$$

$$p^1: \quad \ddot{u}_1 + \omega^2 u_1 + \gamma_1 u_0 + \delta_1 \alpha_2 u_0^3 = 0. \tag{70}$$

Considering Eq. (69) with initial conditions $u(0) = A$ and $\dot{u}(0) = 0$ gives $u_0 = A\cos(\omega t)$. Substituting u_0 into Eq. (70), we obtain:

$$p^1: \quad \ddot{u}_1 + \omega^2 u_1 + \gamma_1 A\cos(\omega t) + \delta_1 \alpha_2 A^3 \cos^3(\omega t) = 0. \tag{71}$$

Using the following Fourier series expansion, we have:

$$\delta_1 \alpha_2 A^3 \cos^3(\omega t) =$$
$$= \sum_{n=0}^{\infty} a_{2n+1} \cos\left[(2n+1)\omega t\right] = \left(\alpha_1 \cos(\omega t) + ...\right) =$$
$$= \left(\frac{4}{\pi}\delta_1 \alpha_2 A^3 \int_0^{\pi/2}\left(\cos^4(\varphi)\right)d\varphi\right)\cos(\omega t) + ... =$$
$$= \frac{3A^3 \alpha_2 \delta_1}{4}. \tag{72}$$

Substituting Eq. (72) into Eq. (71) gives:

$$\ddot{u}_1+\omega^2 u_1+\left(\gamma_1 A+\frac{3A^3\alpha_2\delta_1}{4}\right)cos(\omega t)+$$
$$+\sum_{n=1}^{\infty}a_{2n+1}\,cos\left[(2n+1)\omega t\right]=0. \quad (73)$$

No secular term in u_1 requires that:

$$\gamma_1 A+\frac{3A^3\alpha_2\delta_1}{4}=0\,. \quad (74)$$

Setting p = 1 in Eqs. (25) and (26), we have:

$$\alpha_1=\omega^2+\gamma_1\,; \quad (75)$$

$$1=\delta_1\,. \quad (76)$$

From Eqs. (75) and (76), we obtain:

$$\omega=\sqrt{\alpha_1+\frac{3}{4}\alpha_3 A^2}\;; \quad (77)$$

$$T=\frac{4\pi}{\sqrt{4\alpha_1+3\alpha_3 A^2}}\,. \quad (78)$$

6. Results and discussions

As it is apparent in section 4, the periodic solutions of IPM, VA, and PEM for a current-carrying conductor with cubic non-linearity are equal. In order to, substitute $\alpha_1=1-\Delta\big/(1-\alpha)^2$, $\alpha_2=-\Delta\big/(1-\alpha)^3$ and $\alpha_3=-\Delta\big/(1-\alpha)^4$ into results of periodic solutions for example Eqs. (56) and (57) , the frequency and period values of Eq. (33) can be written as follow:

$$\omega=\sqrt{\frac{12\left(1-\Delta\big/(1-\alpha)^2\right)\pi+32\left(-\Delta\big/(1-\alpha)^3\right)A+9\left(-\Delta\big/(1-\alpha)^4\right)A^2\pi}{12\pi}}\;; \quad (79)$$

$$T=\frac{4\sqrt{3}\pi^{3/2}}{\sqrt{12\left(1-\Delta\big/(1-\alpha)^2\right)\pi+32\left(-\Delta\big/(1-\alpha)^3\right)A+9\left(-\Delta\big/(1-\alpha)^4\right)A^2\pi}}\,. \quad (80)$$

Similarly, substituting $\alpha_1=\left(k_1/m-2P/l\,m\right)$ and $\alpha_2=\left(k_3/m-2P/l^3\,m\right)$ into Eqs. (77) and (78) gives the following frequency and period values for Eq. (58):

$$\omega=\sqrt{\left(\frac{k_1}{m}-\frac{2P}{l\,m}\right)+\frac{3}{4}\left(\frac{k_3}{m}-\frac{2P}{l^3 m}\right)A^2}\;; \quad (81)$$

$$T=\frac{4\pi}{\sqrt{4\left(\frac{k_1}{m}-\frac{2P}{l\,m}\right)+3\left(\frac{k_3}{m}-\frac{2P}{l^3 m}\right)A^2}}\,. \quad (82)$$

6.1. Analytical solutions of current-carrying wire conductor equation

In this section, we compare the analytical approximate periods of Eq. (33) with the exact ones. Considering [25], the exact solution of Eq. (33) is expressed in appendix A.

Using Δ = –3/4, α = –1/2 and Δ = 1/8, $\alpha=\left(2-\sqrt{2}\right)\big/4$, the exact period T_e [25] and approximate periods T_{IPM}, T_{VA}, and T_{EPM} are listed in Table 1. From Table 1 we can obtain that the presented approximate solutions are excellent for all permitted oscillation amplitudes. In general, the first approximate periods of IPM, VA, and PEM are acceptable.

Comparisons of the approximate analytical solution with the exact solutions for given Δ = –3/4, α = –1/2 and Δ = 1/8, $\alpha=\left(2-\sqrt{2}\right)\big/4$ and different amplitudes of oscillation A are shown in Figs. 3-6, respectively.

Table 1

Comparison of approximate and "exact" periods for current-carrying conductor

A	B[26]	T_e [26]	T_{app}	T_{app}/T_e
a) $\Delta=-3/4$, $\alpha=-1/2$			$T_{IPM}=T_{EPM}=$ $=T_{VA}$	
0.1	0.10112	5.43974	5.40109	0.99289
0.4	0.41917	5.41309	5.26059	0.97183
0.7	0.76545	5.34444	5.09664	0.95363
1.0	1.16058	5.20343	4.91674	0.94490
1.2	1.48388	5.02518	4.79132	0.95346
1.4	1.97647	4.65034	4.66357	1.00284
1.43	2.10317	4.54583	4.64431	1.02166
b) $\Delta=1/8$, $\alpha=(2-\sqrt{2})/4$				
0.1	0.09839	6.91230	6.98305	1.01023
0.3	0.28465	6.99470	7.20094	1.02948
0.5	0.44943	7.25765	7.51543	1.03552
0.6	0.51485	7.63026	7.71964	1.01171
0.63	0.53007	7.85127	7.78834	0.99198

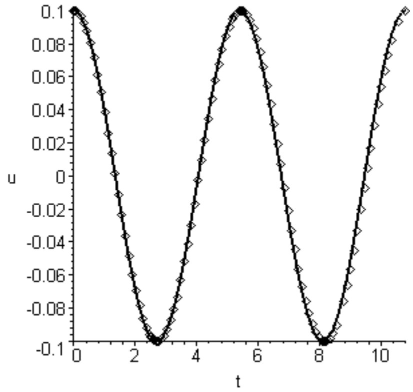

Fig. 3 Comparison of approximate periodic solutions of current-carrying conductor equation (Eq. 34) with the exact one for $\Delta = -3/4$, $\alpha = -1/2$, with $u(0) = 0.1$

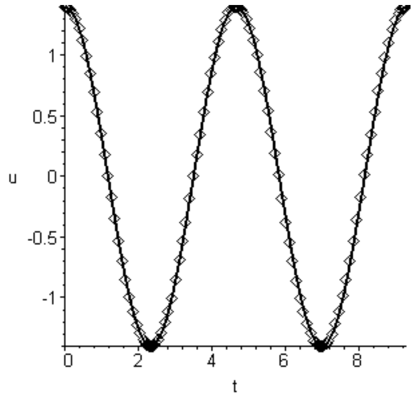

Fig. 4 Comparison of approximate periodic solutions of current-carrying conductor equation (Eq. 34) with the exact one for $\Delta = -3/4$, $\alpha = -1/2$, with $u(0) = 1.4$

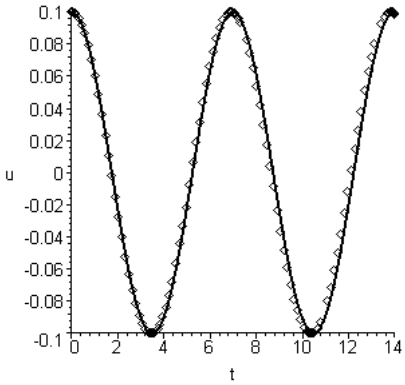

Fig. 5 Comparison of approximate periodic solutions of current-carrying conductor equation (Eq. 34) with the exact one for $\Delta = 1/8$, $\alpha = \left(2-\sqrt{2}\right)/4$ with $u(0) = 0.1$

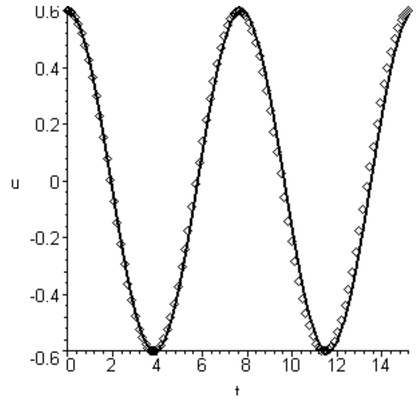

Fig. 6 Comparison of approximate periodic solutions of current-carrying conductor equation (Eq. 34) with the exact one for $\Delta = 1/8$, $\alpha = \left(2-\sqrt{2}\right)/4$ with $u(0) = 0.6$

Solid line: Exact solution. Diamond symbol: Approximate IPM, VA and PEM.

6.2. Analytical solutions of bucking of a column equation

The exact frequency ω_e for a dynamic system governed by Eq. (58) is presented in Appendix B. As we know Eq. (58) is similar to a type of Duffing equation. So, the maximum amplitude A of the oscillation satisfies $\alpha_2 A^2 = -\alpha_1$; the Duffing equation has a heteroclinic orbit with period $+\infty$ [26]. Hence, in order to avoid the heteroclinic orbit with period $+\infty$, the value of k_3 in the bucking of a column equation should satisfy the following equation:

$$k_3 > \frac{-k_1}{A^2} + \frac{2p}{l}\left(\frac{1}{A^2} + \frac{1}{l^2}\right), \tag{83}$$

where $k_1, l \in \mathrm{R}^+$ and $A, p \in R$.

To further illustrate and verify the accuracy of the proposed analytical approaches for Eq. (58), the corresponding comparisons of analytical solutions with exact results for specific parameters and initial values consisting m, p, l, k_1, k_2 and A are tabulated in Table 2.

Table 2

Comparison of approximate and "exact" periods for the bucking of a column

Constant parameters							T_e	T_{app}	T_{app}/T_e
m	L	P	k_1	k_3	A			$T_{IPM} = T_{EBM} = T_{VA}$	
1	1	1	10	5	1		1.96451	1.96254	1.00101
5	1.5	5	5	6	3		3.32368	3.23744	1.02664
10	10	10	10	50	10		0.33143	0.32426	1.02212
50	25	40	30	100	20		0.26208	0.25640	1.02216
70	20	-30	50	100	10		0.30993	0.30323	1.02212
100	50	150	70	20	100		0.16580	0.162206	1.02218
500	150	220	120	500	0.5		9.71672	9.676370	1.00417
1000	500	1000	500	500	1		6.75871	6.73241	1.00391

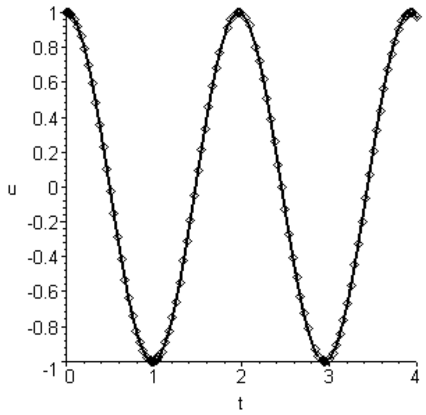

Fig. 7 Comparison of approximate periodic solutions of Bucking of a Column equation (Eq. 58**)** with the exact one for $m = l = p = 1$, $k_1 = 10$, $k_3 = 5$ with $u(0) = 1$

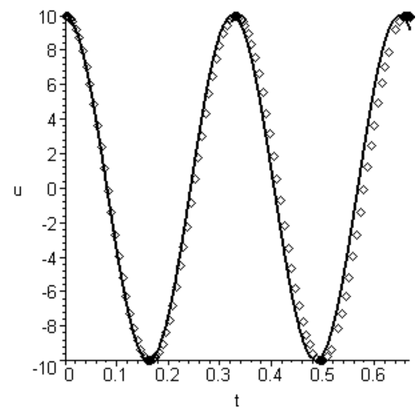

Fig. 8 Comparison of approximate periodic solutions of Bucking of a Column equation (Eq. 58**)** with the exact one for $m = l = p = k_1 = 10$, $k_3 = 50$ with $u(0) = 10$

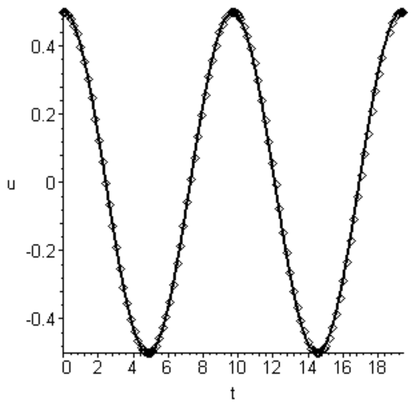

Fig. 9 Comparison of approximate periodic solutions of Bucking of a Column equation (Eq. 58**)** with the exact one for $m = 500$, $l = 150$, $p = 220$, $k_1 = 120$, $k_3 = 500$ with $u(0) = 0.5$

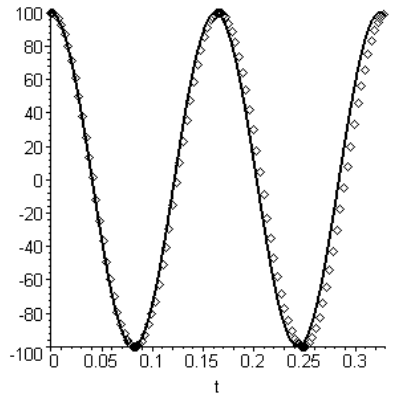

Fig. 10 Comparison of approximate periodic solutions of Bucking of a Column equation (Eq. 58) with the exact one for $m = 100$, $l = 50$, $p = 150$, $k_1 = 70$, $k_3 = 20$ with $u(0) = 100$

Solid line: Exact solution. Diamond symbol: Approximate IPM, VA and PEM.

Figs. 7-10 indicate the comparison of these analytical methods for different parameters with initial values which are in an excellent agreement with the exact solutions.

Of course the accuracy of these methods can be improved upon using higher–order approximate solutions for approximations methods. Hence, it is concluded for providing an excellent agreement with exact solutions for the nonlinear Duffing equation.

7. Conclusions

In summary, three analytical approximations to the periodic solution of SDOF systems including current-carrying conductor and bucking of a column are constructed using IPM, PEM, and VA approaches. According to the results (Tables. 1-2, and Figs. 3-10), we can see that the presented approximate results are absolutely equal and differences between analytical and exact solutions are negligible. In other words, the first–order approximate solutions of IPM, EBM, and VA benefit a high accuracy and the percentage error improves significantly from lower–order to higher–order analytical approximations for different parameters and initial amplitudes.

Appendix A

For achieving the exact period T_e of Eq. (1), substituting a new independent variable $u = x - \alpha$ into Eq. (1) leads to [25]:

$$\ddot{u} + \alpha + u - \frac{\Delta}{1 - \alpha - u} = 0,\ u(0) = A,\ \dot{u}(0) = 0\,, \quad (A.1)$$

where α is one of the stable equilibrium points and $A = \tilde{A} - \alpha$. The corresponding potential energy function is:

$$V(u) = \frac{1}{2}(u + \alpha)^2 + \Delta ln|1 - u - \alpha|\,. \quad (A.2)$$

And it reaches its minimum at $u = 0$. Thus, the system will oscillate between asymmetric limits $[-B, A]$ where both $-B$ ($B > 0$) and A have the same energy level, i.e.:

$$V(-B) = V(A)\,. \quad (A.3)$$

The exact period $T_e(A)$ is:

$$T_e(A) = 2\int_{-B}^{A} \begin{bmatrix} (A+\alpha)^2 - (u+\alpha)^2 + \\ +2\Delta ln\left|\dfrac{1-A-\alpha}{1-u-\alpha}\right| \end{bmatrix}^{-1/2} du \, , \qquad \text{(A.4)}$$

where B is given by Eqs. (A.2) and (A.3).

Appendix B

The exact solution of Eq. (58) can be obtained by integrating the governing differential equation and imposing the initial conditions in Eq. (58) as follows:

$$\frac{1}{2}\dot{v}^2 + \frac{\alpha}{2}v^2 + \frac{\beta}{2}v^4 = C, \qquad \forall t \, , \qquad \text{(B.1)}$$

which C is a constant. Imposing initial conditions in Eq. (58) yields:

$$C = \frac{\alpha}{2}A^2 + \frac{\beta}{4}A^4 \, . \qquad \text{(B.2)}$$

Equating Eqs. (B.1) and (B.2) yields:

$$\frac{1}{2}\dot{v}^2 + \frac{\alpha}{2}v^2 + \frac{\beta}{4}v^4 = \frac{\alpha}{2}A^2 + \frac{\beta}{4}A^4 \qquad \text{(B.3)}$$

or equivalently

$$dt = \frac{dv}{\sqrt{\alpha\left(A^2 - v^2\right) + \dfrac{\beta}{2}\left(A^4 - v^4\right)}} \, . \qquad \text{(B.4)}$$

Integrating Eq. (B.4), the period of oscillation T_e is:

$$T(A) = 4\int_0^A \frac{dv}{\sqrt{\alpha\left(A^2-v^2\right)+\dfrac{\beta}{2}\left(A^4-v^4\right)}} \, . \qquad \text{(B.5)}$$

Substituting $v = A cos t$ into Eq. (B.5) and integrating:

$$T(A) = \frac{4}{\sqrt{\alpha+\beta A^2}}\int_0^{\pi/2} \frac{dt}{\sqrt{1-\delta \, sin^2 t}} \, , \qquad \text{(B.6)}$$

which

$$\delta = \frac{\beta A^2}{2\left(\alpha + \beta A^2\right)} \, . \qquad \text{(B.7)}$$

The exact frequency ω_e is also a function of A and can be obtained from the period of the oscillation as:

$$\omega_e(A) = \frac{\pi\sqrt{\alpha+\beta A^2}}{2}\left(\int_0^{\pi/2} \frac{dt}{1-\delta \, sin^2 t}\right)^{-1} . \qquad \text{(B.8)}$$

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SROVĖS PERDAVIMO LAIDŲ IR ATRAMŲ VIRPESIŲ TARPUSAVIO SĄVEIKA

R e z i u m ė

Straipsnyje aptariami apytiksliai analitiniai metodai – iteracijų trikdžio metodas (ITM), parametrų išplėtimo metodas (PIM) ir variacinio priartėjimo metodas (VPM) – vieno laisvės laipsnio (1 l.l.) netiesinėms virpesių sistemoms tirti. Pateikta keletas skaitinių pavyzdžių, tokių kaip dinaminės elektros srovės laidų ir kolonų virpesių analizė ir jos rezultatų palyginimas su tiksliais duomenimis. Tiriama skirtingų specifinių parametrų ir pradinių reikšmių, pavyzdžiui, masės ir standumo, įtaka ir pasiekiamas norimas tikslumas – išskirtinis pasiūlytiems sprendimams visame virpesių amplitudžių diapazone.

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ON THE BEHAVIOUR OF CURRENT-CARRYING WIRE-CONDUCTORS AND BUCKING OF A COLUMN

S u m m a r y

This paper applies approximate analytical methods namely Iteration Perturbation Method (IPM), variational approach (VA) and Parameter Expanding Method (PEM) to Single-Degree-Of-Freedom (SDOF) nonlinear oscillation systems. Some numerical cases as dynamic behavior of current-carrying wire-conductors and bucking of a column as well as their comparisons with the exact solutions are presented. Different specific parameters and initial values comprising the mass and stiffness are studied within the current research and excellent accuracy which is the most significant feature of the proposed solutions, is reported for the whole range of oscillation amplitude values.

Keywords: Nonlinear oscillation; Current-carrying conductor; Bucking of a column; Iteration Perturbation Method (IPM); Parameter Expanding Method (PEM); Variational Approach (VA).

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