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Mathematical modeling of variable lead helix and design of transition curve in VLSM

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1. Introduction

The traditional ball screw can only perform in helical transmission with constant lead, and the rotational motion of screw is linear with the linear motion of nut. In order to get nonlinear relation between the two motions, motor control is used. The variable lead screw mechanism (VLSM) is generally used to change linear motion to angular motion or vice versa for a prescribed nonlinear input-output relationship. There are some different types of VLSM discussed in previous papers. Transmission quality of VLSM is evaluated by Ming J. Tsai [1]. Yan and Cheng studied several kinds of meshing rollers with concave and convex involute surfaces. Using conjugate surface theory, Liu and Yan [2] studied the surface geometry of the screw in the VLSM and the basic equations were derived for machine tool settings used in manufacturing the screw.

Transition curve design is important to transmission performance. Rolling-ball method was proposed by Rossignac [3] to structure transition surface, which simulate a ball rolling along the intersecting line of two surfaces. Choi B. K., Harada and Farouki [4] then studied the constant and variable radius methods. This paper researches on the geometrical property of spiral line in cylindrical surface which is helpful to the variable lead transmission performance.

2. Testing procedures

The helical character can be described with lead p or lead angle λ. Considering of the variation of revolver radius with generatrix, an arbitrary revolver has to be described with cylindrical coordinate (r, θ, z). Lead angle λ is the function of z. The general spiral line is represent by the equation $l=\left(r(z)cos\theta(z), r(z)sint\theta(z), z\right)$.

Fig. 1 shows the general spiral line in Cartesian coordinate system. P is a moving point on spiral line whose speed vector $\boldsymbol{V}$ can be resolved along three directions, which are axial vector $\boldsymbol{V}_a$, circumferential vector $\boldsymbol{V}_t$ and radial vector $\boldsymbol{V}_r$.

The speed vector of revolver generatrix tangent direction is $\boldsymbol{V}_u=\boldsymbol{V}_a+\boldsymbol{V}_r$. The lead angle is expressed by speed vector as:

$$tan\lambda(z)=\left|\boldsymbol{V}_u(z)\right|/\left|\boldsymbol{V}_t(z)\right|. \tag{1}$$

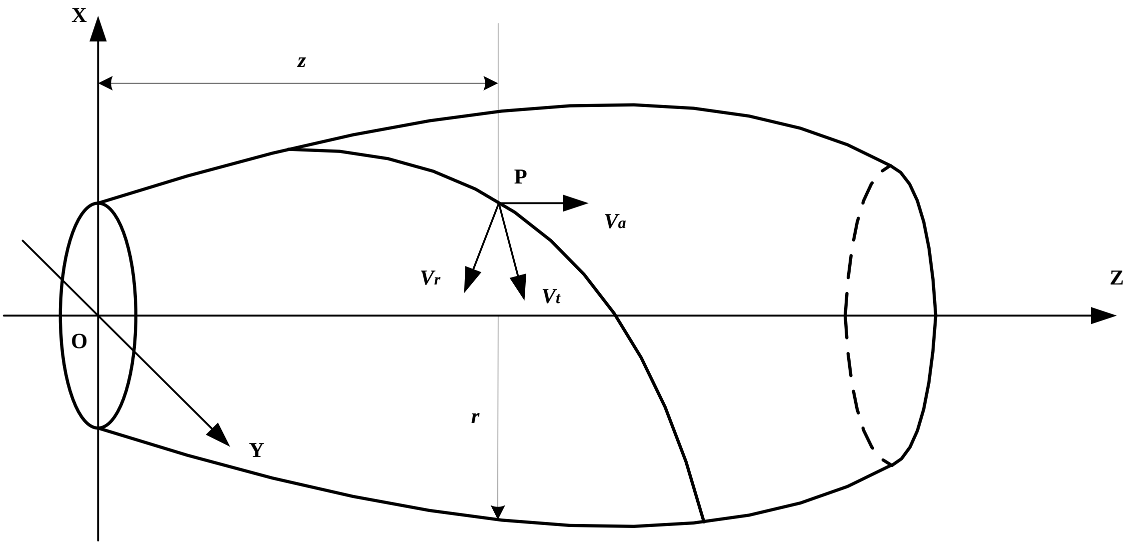

Fig. 1 Coordinate of variable radius and lead spiral line

The circumferential speed is:

$\boldsymbol{V}_t(z)=\omega r(z)=\dfrac{d\theta}{dt}r(z)$, and easily we can get:

$$d\theta=\omega dt=\boldsymbol{V}_t(z)dt/r(z). \tag{2}$$

According to equation above and $\boldsymbol{V}_a=dz/dt$, we can $\dfrac{d\theta}{dz}=\dfrac{V_t(z)dt}{V_a(z)r(z)dt}=\dfrac{1}{r(z)tan\lambda(z)}\sqrt{1+r'^2}$,where r' is dr/dz.

The function $\theta(z)$ is expressed as follows:

$$\theta(z)=\int\frac{1}{r(z)tan\lambda(z)}\sqrt{1+r'^2}dz\,, \tag{3}$$

that the local shape of variable lead helix is described by $\theta(z)$, $\lambda(z)$ and $r(z)$ exactly.

3. Curvature of variable lead helix

It is convenient to get anti-function of $\theta=\theta(z)$. Here, the general helix parameter equation $l(t)$ is represent below:

$$\begin{cases} x=r\big(z(\theta)\big)cos\theta, \\ y=r\big(z(\theta)\big)sin\theta, \\ z=z(\theta). \end{cases}$$

If θ is replaced by parameter t, the equation is given as:

$$\begin{cases} x=r\big(f(t)\big)cos\,t, \\ y=r\big(f(t)\big)sin\,t, \\ z=f(t), \end{cases} \tag{4}$$

so the first-order l', second-order l'' and third-order l''' derivative forms are respectively expressed as follows:

$$\begin{cases} x'=r'f'cos\,t-rsin\,t, \\ y'=r'f'sin\,t+rcos\,t, \\ z'=f'. \end{cases} \tag{5}$$

$$\begin{cases} x''=(r''f'+r'f''-r)cos\,t-(f'+1)r'sin\,t, \\ y''=(r''f'+r'f''-r)sin\,t+(f'+1)r'cos\,t, \\ z''=f''. \end{cases} \tag{6}$$

$$\begin{cases} x'''=(r'''f'+2r''f''+r'f'''-2r'-r'f')cos\,t- \\ \quad -(2r''f'+2r'f''-r+r'')sin\,t, \\ y'''=(r'''f'+2r''f''+r'f'''-2r'-r'f')sin\,t+ \\ \quad +(2r''f'+2r'f''-r+r'')cos\,t, \\ z'''=f'''. \end{cases} \tag{7}$$

The curvature and torsion of moving point P are obtained when plugging Eqs. (5)-(7) into Eqs. (8) and (9):

$$\kappa=|l'\times l''|/|l'|^3\,; \tag{8}$$

$$\tau=(l',l'',l''')/|l'\times l''|^2\,. \tag{9}$$

From the equation above, we know that the curvature and torsion expression of general spiral line is too complicated. At present we are not interested in that how the curvature is affected by variable radius, but that how the curvature is affected by variable lead. Now the revolver is considered as a cylinder with constant radius a, and the lead just relates to the function $f(t)$.

The curvature and torsion formula are easily derived as follows:

$$\kappa=\frac{a\left(f''^2+f'^2+a^2\right)^{1/2}}{\left(a^2+f'^2\right)^{3/2}}\,; \tag{10}$$

$$\tau=\frac{f'+f'''}{f''^2+f'^2+a^2}\,, \tag{11}$$

where a is the constant radius of cylinder.

4. Normal and geodesic curvature of variable lead helix

The curvature vector of the moving point P could be resolved into normal vector and geodesic vector, and they are perpendicular, thus $\kappa=d\boldsymbol{\alpha}/ds=\kappa_n\boldsymbol{n}+\kappa_g\boldsymbol{\alpha}\ \boldsymbol{n}$, where $\boldsymbol{\alpha}$ is the unit vector of point P along the tangent direction, and $\boldsymbol{n}$ is the unit vector of point P along normal direction. κ_n is normal curvature while κ_g is geodesic curvature, then:

$$\kappa^2=\kappa_n{}^2+\kappa_g{}^2\,. \tag{12}$$

According to Meusnier theorem, the curvature of P on the helix is equal to the latitude circle curvature.

The cylindrical surface parameter equation is $S(t,z)=(a\,cos\,t,\,a\,sin\,t,\,z)$, then the first fundamental form and second fundamental form are:

$$\begin{cases} I=a^2dt^2+dz^2, \\ II=-adt^2. \end{cases} \tag{13}$$

then the helix normal curvature is:

$$|\boldsymbol{\kappa}_n|=\frac{II}{I}=\frac{-adt^2}{a^2dt^2+dz^2}=\frac{-a}{a^2+f'^2}\,, \tag{14}$$

where a is the constant radius.

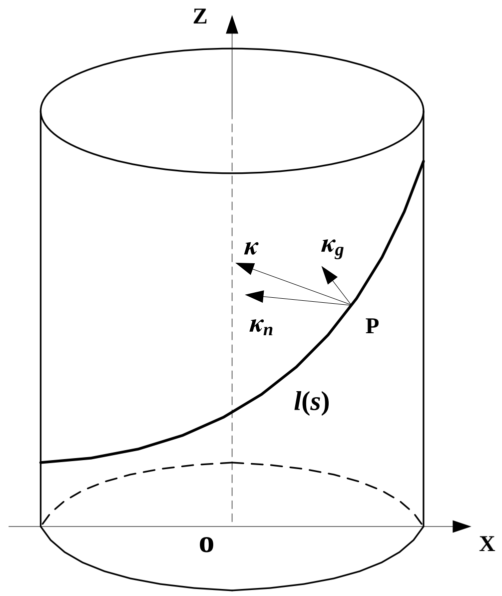

Fig. 2 Normal and geometric curvature of helix

According to equation $\kappa^2 = \kappa_n{}^2 + \kappa_g{}^2$ and Eq. (10), the geometric curvature equation is:

$$\kappa_g = \frac{a|f''|}{\left(a^2 + f'^2\right)^{3/2}}. \tag{15}$$

We can conclude that the normal curvature is determined by the radius a and first-order derivative of function $f(t)$, while the geometric curvature relates to the radius a and second-order derivative of function $f(t)$. If $f(t)$ is a linear function of t, then $\kappa_g \equiv 0$, thus the curve is a geodesic curve.

<u>Corollary</u>: The geometric curvature of the helix on the cylindrical surface is equal to the geometric curvature of plane line which is obtained by expanding the cylindrical surface along axial direction.

<u>Proof</u>: The cylindrical surface is expanded along axial direction, and the helix change to a plane line with the parameter equation $P(t) = (at, f(t), 0)$. Easily, the first-order derivative of $P(t)$ is $P'(t) = (a, f', 0)$, and the second-order derivative of $P(t)$ is $P''(t) = (0, f'', 0)$.

According to Eq. (8), the curvature of the plane curve is shown below:

$$\kappa = \frac{a|f''|}{\left(a^2 + f'^2\right)^{3/2}} \tag{16}$$

and the corollary above is proved.

5. Design of transition curve on cylindrical surface

According to the corollary above, the space curve can be researched in plane which is simplified. The transition curve needs to meet some conditions according to the specific circumstances.

VLSM here is a variable lead screw in rail door system. Usually the door speed of opening and closing is designed quickly and smoothly for the convenience of passengers, so the transition curve should be first-order continuous at the two connection point at least. Circular arc can meet the condition, besides it has n-order continuous characteristics with simple equation, thus, we try to connect two curves in cylindrical surface with circular arc.

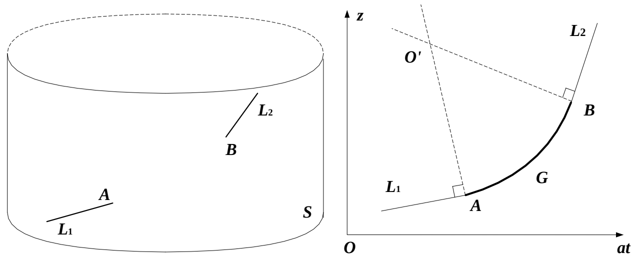

Fig. 3 Transition curve of circular arc

Fig. 3 shows two curves L_1 and L_2 in cylindrical surface with unequal slopes at end point A and B. The local coordinate is established which its horizontal ordinate is at. The two curves can be regarded as constant slope curves. L_1 and L_2 become to straight line when the cylindrical surface is expanded. Curve G is the circular arc that connects L_1 and L_2. The coordinate of A is $A(at_A, z_A)$ and the slope k_A of A is $p_A/2\pi a$; the coordinate of B is $B(at_B, z_B)$, and the slope k_B of A is $p_B/2\pi a$, where p_A and p_B are helical pitches. The coordinate of O' is

$$\left(\frac{k_A k_B (z_B - z_A) + a(t_B k_A - t_A k_B)}{k_A - k_B}, \frac{k_A z_A - k_B z_B + a(t_A - t_B)}{k_A - k_B}\right),$$

and the parametric equation of transition circular arc is:

$$G(t) = \begin{cases} x = a\cos t, \\ y = a\sin t, \\ z = \left(k_A z_A - k_B z_B + a(t_A - t_B)\right)/(k_A - k_B) + \\ \quad + \sqrt{R^2 - a^2\left(1 + k_A{}^2\right)(t - t_A)^2}, \end{cases}$$

where $t_A \le t \le t_B$.

Polynomial curve is used in curve fitting problem generally, and the polynomial curve is discussed in plane as before. The plane equation is:

$$G(t) = \begin{cases} x = at, \\ z = f(t), \end{cases}$$

where $t_A \le t \le t_B$.

The boundary conditions are shown as follows:

$$\begin{cases} x(t_A) = at_A, \\ z(t_A) = z_A, \end{cases} \qquad \begin{cases} x(t_A) = at_B, \\ z(t_B) = z_B, \end{cases}$$

$$\begin{cases} x'(t_A) = a, \\ z'(t_A) = p_A/2\pi, \end{cases} \qquad \begin{cases} x'(t_B) = a, \\ z'(t_B) = p_B/2\pi. \end{cases}$$

Because of the uniqueness condition, the function $f(t)$ is defined as:

$$f(t, \beta) = \beta t^4 + m_1 t^3 + m_2 t^2 + m_3 t + m_4, \tag{17}$$

Where: m_i is an undetermined coefficient and β is shape parameter.

Putting the boundary conditions into Eq. (14), we obtained linear simultaneous equation:

$$\begin{bmatrix} \beta t_A{}^4 & t_A{}^3 & t_A{}^2 & t_A & 1 \\ \beta t_B{}^4 & t_B{}^3 & t_B{}^2 & t_B & 1 \\ 4\beta t_A{}^3 & 3t_A{}^2 & 2t_A & 1 & 0 \\ \beta t_B{}^3 & 3t_B{}^2 & 2t_B & 1 & 0 \end{bmatrix} \begin{bmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \end{bmatrix} = \begin{bmatrix} z_A \\ z_B \\ p_A/2\pi \\ p_B/2\pi \end{bmatrix}, \tag{18}$$

where β determines m_i to obtain better transition curve.

In the system of railway traffic, the train door system uses VLSM to realize the self-locking with zero lead at the end of screw. Fig. 4 shows the working principle of self-locking in the VLSM. Z direction is the screw axis direction. We consider the screw coordinate system as static coordinate, then the roller which connected with nut rolls along the curve $L_1 - G - L_2$. When the roller enters into the region L_2, the VLSM can not change the straight line motion of nut to rotary motion of screw.

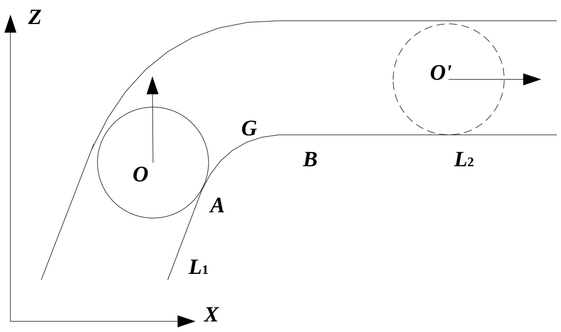

Fig. 4 Self-locking principle of VLSM

Sometimes the transition curves discussed above can not meet other conditions such as slickness, fairness, and then the functional of transition curve is built as Eq. (19) according to constraint condition to solve the extreme-value problem:

$$J\left[z(t)\right]=\int_{t_A}^{t_B}F\left(z,z',z'',...\right)dt\ . \tag{19}$$

This is n-order derivative unary variation problem which should satisfy Euler-Poisson equation:

$$\sum_{k=0}^{n}(-1)^k\frac{d^k}{dx^k}F_{z^{(k)}}=0\ , \tag{20}$$

where F has n+2 order continuous derivative; z has n order continuous derivative and need 2n boundary condition to be determined.

6. Conclusions

1. This paper firstly established the mathematical model of variable lead helix in rotary surface and discussed the geometrical characteristic of helix in cylindrical surface. We got the general expression of curvature and torsion in cylindrical surface refer to Eqs. (10) and (11).

2. The geometic curvature of the helix on the cylindrical surface is equal to the geometric curvature of plane line which is obtained by expanding the cylindrical surface along axial direction. The geometric curvature is $\kappa=\frac{a\left|f''\right|}{\left(a^2+f'^2\right)^{3/2}}$.

3. The transition circular arc equation is shown as Eq. (13), and the polynomial curve can satisfy the boundary condition well. Shape parameter method is appropriate for optimization problem and variation method is appropriate for the extreme value problem.

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MATEMATINIS VARANČIOJO SRAIGTO MODELIAVIMAS IR KVSM PEREINAMOSIOS KREIVĖS PROJEKTAVIMAS

R e z i u m ė

Straipsnyje analizuojamas kintamo varančiojo sraigto mechanizmo (KVSM) kintamo spindulio varančiojo sraigto matematinis modelis, sukurtas taikant kintamos varančiosios transmisijos teorijos metodą. Jame taip pat studijuojamos kintamo varančiojo sraigto cilindrinio paviršiaus geometrinės charakteristikos, o kreivumas ir susukimas yra išreikšti $z=f(t)$ funkcija, kuri tinka pereinamajai kreivei tirti. Prieita prie išvados, kad sraigto geometrinis kreivumas cilindriniame paviršiuje yra tolygus geometriniam kreivumui linijos plokštumoje, kuri gaunama ištęsiant cilindrinį paviršių ašies kryptimi, kad pereinamosios kreivės brėžinį būtų galima perkelti į plokštumą.

Pereinamosios kreivės brėžinyje apskritiminis lankas ir polinominė kreivė buvo pasirinkti sujungiant dvi žinomas kreives, paskui, remiantis kraštinėmis sąlygomis, įterptas pereinamosios kreivės funkcionalas.

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MATHEMATICAL MODELING OF VARIABLE LEAD HELIX AND DESIGN OF TRANSITION CURVE IN VLSM

S u m m a r y

This paper established the general mathematical model of variable radius variable lead helix, which applies the theory method for variable lead transmission. It also studeid the geometrical characteristics of variable lead helix in cylindrical surface, and the curvature and torsion is expressed by function $z=f\left(t\right)$, which is good to the research of transition curve. The corollary is put forward and explained that the geometric curvature of the helix on the cylindrical surface is equal to the geometric curvature of plane line which is obtained by expanding the cylindrical surface along axial direction, that transition curve design can be carried out in the plane. In the design of transition curve, circular arc and polynomial curve were chosen to connect two known curves respectively, and then according to the extreme conditions the transition curve functional is built up.

Keywords: variable led helix, transition curve.

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