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Influence of side walls on the oscillating motion of a Maxwell fluid over an infinite plate

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1. Introduction

Flows of non-Newtonian fluids are important due to their applications in various branches of industry and technology and their study presents a special challenge to engineers, physicists and mathematicians. The motion of non-Newtonian fluids due to the oscillations of a plate has been studied by many authors while exact solutions for motions induced by an infinite plate that applies an oscillating shear stress to the fluid are almost absent. However, to the best of our knowledge, the first closed-form starting solutions for flows of Newtonian fluids due to cosine and sine oscillations of a flat plate have been established after a long time by Erdogan [1]. Recently, similar solutions for the motion of the same fluids due to an infinite plate that applies oscillating shear stresses to the fluid have been established in [2]. These solutions have been obtained as limiting cases of more general solutions corresponding to the unsteady motion of the fluid between two side walls perpendicular to a flat plate that applies oscillating shear stresses to the fluid.

The aim of this note is to extent the results from [2] to a larger class of fluids. More exactly, our interest is to provide close-form expressions for the starting solutions corresponding to the unsteady motion of a Maxwell fluid due to an infinite plate that applies oscillating shear stresses to the fluid. For generality, the general solutions will be established for the motion of the fluid between two parallel walls perpendicular to the plate. In the absence of the side walls, namely when the distance between walls tends to infinity, these solutions aspire to the similar solutions corresponding to the motion over an infinite plate. Furthermore, if the relaxation time $\lambda \to 0$, the solutions that have been obtained tend to the known solutions for Newtonian fluids. Such solutions are uncommon in the literature because, unlike the usual no slip condition, a boundary condition on the shear stress is used. This is very important as in some problems, what is specified is the force applied on the boundary.

It is also important to bear in mind that the "no slip" boundary condition may not be necessarily applicable to flows of polymeric fluids that can slip or slide on the boundary. Thus, the shear stress boundary condition is particularly meaningful. To the best of our knowledge, the first exact solutions for motions of non-Newtonian fluids in which the shear stress is given on the boundary are those of Waters and King [3] for Oldroyd-B fluids and Bandelli et. al [4] for second grade fluids. Meanwhile, other exact solutions for different motions of viscous or non-Newtonian fluids have been established [5-12].

The present solutions, as well as those obtained in [2], are written as a sum of steady-state and transient solutions. They describe the motion of the fluid some time after its initiation. After that time, when the transients disappear, the fluid is moving according to the steady-state solutions which are periodic in time and independent of initial conditions. However, they satisfy the governing equations and boundary conditions. Finally, the distance between walls for which the velocity of the fluid in the middle of the channel is unaffected by their presence and the required time to reach the steady-state are graphically determined. Furthermore, on the basis of an immediate consequence of the governing equations, an important relation with the motion over a moving plate is brought to light.

2. Governing equations

The Cauchy stress tensor $\boldsymbol{T}$ for an incompressible Maxwell fluid is related to the fluid motion in the following manner [13]:

$$\boldsymbol{T} = -p\boldsymbol{I} + \boldsymbol{S}, \;\; \boldsymbol{S} + \lambda\left(\dot{\boldsymbol{S}} - \boldsymbol{LS} - \boldsymbol{SL}^T\right) = \mu\boldsymbol{A}, \tag{1}$$

where: $-p\boldsymbol{I}$ is the indeterminate part of the stress due to the constraint of incompressibility; $\boldsymbol{S}$ is the extra-stress tensor; λ is the relaxation time; $\boldsymbol{L}$ is the velocity gradient; $\boldsymbol{A} = \boldsymbol{L} + \boldsymbol{L}^T$ is the first Rivlin-Ericksen tensor; μ is the dynamic viscosity and the superposed dot denotes the material time derivative. In the following we shall seek a velocity field $\boldsymbol{v}$ and an extra-stress $\boldsymbol{S}$ of the form [14]:

$$\boldsymbol{v} = \boldsymbol{v}(y,z,t) = u(y,z,t)\boldsymbol{i}, \quad \boldsymbol{S} = \boldsymbol{S}(y,z,t), \tag{2}$$

where: $\boldsymbol{i}$ is the unit vector along the x-direction of the Cartesian coordinate system x, y and z. For such flows the constraint of incompressibility is automatically satisfied. If the fluid is at rest at the moment $t = 0$ then:

$$\boldsymbol{v}(y,z,0) = 0, \quad \boldsymbol{S}(y,z,0) = 0 \tag{3}$$

and the second constitutive Eq. (1) leads to the meaningful relations:

$$\left(1+\lambda\frac{\partial}{\partial t}\right)\tau_1(y,z,t)=\mu\frac{\partial u(y,z,t)}{\partial y},\quad \left(1+\lambda\frac{\partial}{\partial t}\right)\tau_2(y,z,t)=\mu\frac{\partial u(y,z,t)}{\partial z}, \tag{4}$$

where $\tau_1(y,z,t)=S_{xy}(y,z,t)$ and $\tau_2(y,z,t)=S_{xz}(y,z,t)$ are the non-trivial shear stresses. In the absence of a pressure gradient in the flow direction and neglecting body forces, Eq. (4) together with the motion equations lead to the governing equation for velocity [14]:

$$\lambda\frac{\partial^2 u(y,z,t)}{\partial t^2}+\frac{\partial u(y,z,t)}{\partial t}=\nu\left[\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right]u(y,z,t);\ y>0, z\in(0,d),\ t>0, \tag{5}$$

where $\nu=\mu/\rho$ is the kinematic viscosity and ρ is the constant density of the fluid. In the following, the governing Eqs. (4) and (5) together with appropriate initial and boundary conditions will be solved using the Fourier and Laplace transforms.

3. Oscillating motion between two side walls perpendicular to a plate

Let us consider an incompressible Maxwell fluid at rest over an infinite flat plate situated in the (x, z) - plane, and between two side walls situated in the planes $z=0$ and $z=d$. After time $t=0^+$ the plate applies an oscillating shear stress:

$$\tau_1(0,z,t)=\frac{f\lambda\omega}{1+(\lambda\omega)^2}\left\{\frac{1}{\lambda\omega}sin(\omega t)-cos(\omega t)+e^{\frac{-t}{\lambda}}\right\}; \tag{6}$$

$$\tau_1(0,z,t)=\frac{f\lambda\omega}{1+(\lambda\omega)^2}\left\{sin(\omega t)+\frac{cos(\omega t)}{\lambda\omega}-\frac{e^{-t/\lambda}}{\lambda\omega}\right\} \tag{7}$$

to the fluid. Here f and $\omega>0$ are constants, ω being the frequency of the shear stress on the boundary. Owing to the shear, the fluid is gradually moved. Its velocity is of the form Eq. $(2)_1$, the governing equations are given by Eqs. (4) and (5) while the appropriate initial and boundary conditions are given by:

$$\left.\begin{aligned}&u(y,z,0)=\frac{\partial u(y,z,0)}{\partial t}=0,\quad \tau_1(y,z,0)=0,\\ &\tau_2(y,z,0)=0;\quad y>0,\ z\in[0,d];\end{aligned}\right\} \tag{8}$$

$$\left.\begin{aligned}&\left(1+\lambda\frac{\partial}{\partial t}\right)\tau_1(0,z,t)=\mu\frac{\partial u(y,z,t)}{\partial y}\Big|_{y=0}=fsin(\omega t)\\ &\text{or } fcos(\omega t);\ z\in(0,d),\ t>0;\end{aligned}\right\} \tag{9}$$

$$u(y,0,t)=u(y,d,t)=0;\ y>0,\ t\ge 0 \tag{10}$$

and (the natural condition of boundedness at infinity):

$$u(y,z,t)\to 0 \text{ as } y\to\infty,\ z\in[0,d],\ t\ge 0. \tag{11}$$

Of course, the expressions of $\tau_1(0,z,t)$ given by Eqs. (6) and (7) are just the solutions of the partial differential Eqs. $(9)_1$ or $(9)_2$. For $\lambda\to 0$, Eqs. (6) and (7) take the simplified forms:

$$\tau_1(0,z,t)=fsin(\omega t) \text{ or } \tau_1(0,z,t)=fcos(\omega t) \tag{12}$$

corresponding to a Newtonian fluid. They are boundary conditions corresponding to the motion of a Newtonian fluid between two side walls perpendicular to a plate that applies an oscillating shear stress $fsin(\omega t)$ or $fcos(\omega t)$ to the fluid. In the following, let us consider the complex fields:

$$\left.\begin{aligned}&V(y,z,t)=u_c(y,z,t)+iu_s(y,z,t);\\ &T_k(y,z,t)=\tau_{kc}(y,z,t)+i\tau_{ks}(y,z,t),\ k=1,2;\end{aligned}\right\} \tag{13}$$

where $u_s(y,z,t)$, $\tau_{ks}(y,z,t)$ and $u_c(y,z,t)$, $\tau_{kc}(y,z,t)$ are the solutions of our problem corresponding to the boundary conditions Eq. (6), respectively Eq. (7), and i is the imaginary unit. In the next, for simplicity, we shall refer to them as to solutions corresponding to the sine and cosine oscillations of the shear stress on the boundary. In view of the above relations, we obtain the following initial-boundary value problem:

$$\left.\begin{aligned}&\lambda\frac{\partial^2 V(y,z,t)}{\partial t^2}+\frac{\partial V(y,z,t)}{\partial t}=\\ &=\nu\left[\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right]V(y,z,t),\\ &y,t>0,\ z\in(0,d);\end{aligned}\right\} \tag{14}$$

$$\left.\begin{aligned}&\left(1+\lambda\frac{\partial}{\partial t}\right)T_1(y,z,t)=\mu\frac{\partial V(y,z,t)}{\partial y},\\ &\left(1+\lambda\frac{\partial}{\partial t}\right)T_2(y,z,t)=\mu\frac{\partial V(y,z,t)}{\partial z},\\ &y,t>0,\ z\in(0,d);\end{aligned}\right\} \tag{15}$$

$$V(y,z,0)=\frac{\partial V(y,z,0)}{\partial t}=0,\ T_k(y,z,0)=0; \tag{16}$$

$$\left(1+\lambda\frac{\partial}{\partial t}\right)T_1(0,z,t)=\mu\frac{\partial V(y,z,t)}{\partial y}\Big|_{y=0}=fe^{i\omega t}; \tag{17}$$

$$V(y,0,t)=V(y,d,t)=0,\ y>0,\ t\ge 0; \tag{18}$$

$$V(y,z,t)\to 0 \text{ as } y\to\infty,\ z\in[0,d],\ t\ge 0. \tag{19}$$

4. Solution of the problem

In order to determine the solution of problem Eqs. (14)-(19) we use the Fourier and Laplace transforms

[15, 16]. Multiplying both sides of Eq. (14) by $\sqrt{2/\pi}\, cos(y\xi)\, sin(\alpha_n z)$, where $\alpha_n = n\pi/d$, integrating with respect to *y* and *z* from 0 to ∞, respectively 0 to d and using the corresponding initial and boundary conditions, we find that:

$$\lambda \frac{\partial^2 V_{cn}(\xi,t)}{\partial t^2} + \frac{\partial V_{cn}(\xi,t)}{\partial t} + \nu\left(\xi^2+\alpha_n^2\right)V_{cn}(\xi,t) = \\ = \sqrt{\frac{2}{\pi}}\frac{f}{\rho}\left[\frac{(-1)^n-1}{\alpha_n}\right]e^{i\omega t}, \tag{20}$$

where the double Fourier cosine and sine transforms:

$$\left.\begin{aligned} &V_{cn}(\xi,t) = \sqrt{\frac{2}{\pi}}\int_0^\infty\int_0^d V(y,z,t)\, cos(y\xi)\, sin(\alpha_n z)\, dz dy,\\ &n = 1,2,3... \end{aligned}\right\} \tag{21}$$

of the function $V(y, z, t)$ have to satisfy the initial conditions:

$$V_{cn}(\xi,0) = \frac{\partial V_{cn}(\xi,0)}{\partial t} = 0\,. \tag{22}$$

Applying the Laplace transform to Eq. (20) we get:

$$\bar{V}_{cn}(\xi,q) = \frac{f}{\rho}\sqrt{\frac{2}{\pi}}\left[\frac{(-1)^n-1}{\alpha_n}\right]\times \\ \times\frac{1}{(q-i\omega)\left[\lambda q^2+q+\nu\left(\xi^2+\alpha_n^2\right)\right]}, \tag{23}$$

where $\bar{V}_{cn}(\xi,q)$ is the Laplace transform of $V_{cn}(\xi,t)$.

Applying the inverse Laplace transform to Eq. (23) we obtain:

$$V_{cn}(\xi,t) = \frac{f}{\rho}\sqrt{\frac{2}{\pi}}\left[\frac{(-1)^n-1}{\alpha_n}\right]F_n(\xi,t); \tag{24}$$

where

$$\left.\begin{aligned} &F_n(\xi,t) = a_n(\xi)\left\{e^{i\omega t} - e^{\frac{-t}{2\lambda}}\left[cosh\left(\frac{c_n(\xi)}{2\lambda}t\right) - \frac{1+2i\omega\lambda}{c_n(\xi)}sinh\left(\frac{c_n(\xi)}{2\lambda}t\right)\right]\right\};\\ &a_n(\xi) = \frac{\left[\nu\left(\xi^2+\alpha_n^2\right)-\lambda\omega^2\right]-i\omega}{\left[\nu\left(\xi^2+\alpha_n^2\right)-\lambda\omega^2\right]^2+\omega^2}\ ,\quad c_n(\xi) = \sqrt{1-4\nu\lambda\left(\xi^2+\alpha_n^2\right)}. \end{aligned}\right\} \tag{25}$$

In the following, we present the solution corresponding to sine oscillations only. For cosine oscillations the solution can be obtained by a similar way. Inverting Eq. (24) by means of the Fourier and Laplace inversion formulae [15, 16] and by using the well known formulae:

$$\int_0^\infty \frac{cos(y\xi)}{\left(\xi^2-b^2\right)^2+c^2}d\xi = \frac{\pi e^{-yB}}{2c\left(A^2+B^2\right)}\left[A cos(yA)+B sin(yA)\right],$$

$$\int_0^\infty \frac{\left(\xi^2-b^2\right)cos(y\xi)}{\left(\xi^2-b^2\right)^2+c^2}d\xi = \frac{\pi e^{-yB}}{2\left(A^2+B^2\right)}\left[B cos(yA)-A sin(yA)\right],\quad 2A^2 = \sqrt{b^4+c^2}+b^2,\quad 2B^2 = \sqrt{b^4+c^2}-b^2,$$

we find the following expression for the velocity $u_s(y,z,t)$:

$$u_s(y,z,t) = \frac{2f}{\mu h}\sum_{n=1}^\infty \frac{(-1)^{n+1}cos(\gamma_m z)}{\gamma_m}\frac{e^{-yB_m}}{\sqrt{A_m^2+B_m^2}}sin\left(\omega t - yA_m + \varphi_m + \frac{\pi}{2}\right) + \\ + \frac{4fe^{\frac{-t}{2\lambda}}}{\mu\pi h}\sum_{n=1}^\infty\frac{(-1)^n cos(\gamma_m z)}{\gamma_m}\int_0^\infty cos(y\xi)\left(\left[\left(\xi^2+\gamma_m^2\right)-\frac{\lambda\omega^2}{\nu}\right]^2+\omega^2\nu^{-2}\right)^{-1}\times \\ \times\left\{\frac{\omega}{\nu}cosh\left(\frac{c_m(\xi)}{2\lambda}t\right) + \frac{\frac{\omega}{\nu}-2\lambda\omega\left[\left(\xi^2+\gamma_m^2\right)-\frac{\lambda\omega^2}{\nu}\right]}{c_m(\xi)}sinh\left(\frac{c_m(\xi)}{2\lambda}t\right)\right\}d\xi,\quad z\in[-h,h], \tag{26}$$

where $2A_m^2 = \sqrt{b_m^4+c^2}+b_m^2,\ 2B_m^2 = \sqrt{b_m^4+c^2}-b_m^2,\ b_m^2 = \frac{\lambda\omega^2}{\nu}-\gamma_m^2,\ c = \frac{\omega}{\nu},\ tan\,\varphi_m = \frac{B_m}{A_m},\ m = 2n-1,\ d = 2h,\ \gamma_m = \frac{m\pi}{2h}.$

Direct computations clearly show that the starting solution

$u_s(y,z,t)$ given by Eq. (26) satisfy all imposed initial and boundary conditions. It is presented as a sum between steady-state and transient solutions and describe the motion of the fluid some time after its initiation. After that time, when the transients disappear, the starting solution tend to the steady-state solution:

$$u_{ss}(y,z,t)=\frac{2f}{\mu h}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}cos(\gamma_m z)}{\gamma_m}\times$$
$$\times\frac{e^{-yB_m}}{\sqrt{A_m^2+B_m^2}}cos(\omega t-yA_m+\varphi_m). \qquad (27)$$

Generally, the starting solutions for unsteady motions of fluids are important for those who want to eliminate the transients from their rheological easurements. Consequently, a very important problem regarding the technical relevance of such solutions is to find the approximate time after which the fluid is moving according to the steady-state solutions. More exactly, in practice, it is necessary to know the required time to reach the steady-state. This time will be later determined by means of graphical illustrations. In order to determine the shear stress $\tau_{1s}(y,z,t)$, for instance, we apply the Laplace transform to Eq. $(15)_1$ and use Eq. (23) to get $\bar{V}(y,z,q)$. Lengthy but straightforward computations lead to:

$$\tau_{1s}(y,z,t)=\frac{2f}{h\sqrt{1+\lambda^2\omega^2}}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}cos(\gamma_m z)}{\gamma_m}\left[e^{-yB_m}sin(\omega t-yA_m-\psi)+\lambda\omega e^{-y\gamma_m-t/\lambda}\left(1+\lambda^2\omega^2\right)^{-1/2}\right]+$$
$$+\frac{4f}{\pi h}e^{\frac{-t}{2\lambda}}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}cos(\gamma_m z)}{\gamma_m}\int_0^{\infty}\frac{\xi\, sin(y\xi)}{\left(\xi^2+\gamma_m^2\right)\left\{\left[\nu\left(\xi^2+\gamma_m^2\right)-\lambda\omega^2\right]^2+\omega^2\right\}}\times$$
$$\times\left\{\omega\left[\nu\left(\xi^2+\gamma_m^2\right)-\lambda\omega^2\right]cosh\left(\frac{c_m(\xi)}{2\lambda}t\right)+\frac{\omega\left[\nu\left(\xi^2+\gamma_m^2\right)+\lambda\omega^2\right]}{c_m(\xi)}sinh\left(\frac{c_m(\xi)}{2\lambda}t\right)\right\}d\xi, \qquad (28)$$

where $tan\psi=\lambda\omega$. The starting solutions Eq. (28) is also presented as a sum of steady-state and transient solutions.

5. Limiting cases

5.1. $\lambda\to 0$ (Newtonian fluids)

By making $\lambda\to 0$ into Eqs. (26)-(28), the solutions [2], Eqs. (14) and (17):

$$u_{sN}(y,z,t)=\frac{2f}{\mu h}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}cos(\gamma_m z)}{\gamma_m}\frac{e^{-yB_{mN}}}{\sqrt{A_{mN}^2+B_{mN}^2}}cos(\omega t-yA_{mN}+\varphi_{mN})+$$
$$+\frac{4f}{\pi\mu h}\left(\frac{\omega}{\nu}\right)\sum_{n=1}^{\infty}\frac{(-1)^{n}cos(\gamma_m z)}{\gamma_m}\int_0^{\infty}\varphi_1(y,\xi)e^{-\nu\left(\xi^2+\gamma_m^2\right)t}d\xi,\quad \varphi_1(y,\xi)=cos(y\xi)\left(\left(\xi^2+\gamma_m^2\right)^2+\omega^2\nu^{-2}\right)^{-1}; \qquad (29)$$

$$\left.\begin{aligned}&\tau_{1sN}(y,z,t)=\frac{2f}{h}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}cos(\gamma_m z)}{\gamma_m}e^{-yB_{mN}}sin(\omega t-yA_{mN})+\\&+\frac{4f}{\mu\pi}\frac{\omega}{\nu}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}cos(\gamma_m z)}{\gamma_m}\int_0^{\infty}\varphi_2(y,\xi)e^{-\nu\left(\xi^2+\gamma_m^2\right)t}d\xi,\quad \varphi_2(y,\xi)=\xi\, sin(y\xi)\left((\xi^2+\gamma_m^2)^2+\omega^2\nu^{-2}\right)^{-1}\end{aligned}\right\} \qquad (30)$$

corresponding to Newtonian fluids performing the same motion are recovered. In these last relations:

$$2A_{mN}^2=\sqrt{\gamma_m^4+c^2}-\gamma_m^2,\ 2B_{mN}^2=\sqrt{\gamma_m^4+c^2}+\gamma_m^2,$$
$$tan\varphi_{mN}=B_{mN}/A_{mN}.$$

5.2. Case $h\to\infty$ (Flow over an infinite plate)

In the absence of side walls, namely when $h\to\infty$, the general solutions Eqs. (26) and (28) corresponding to Maxwell fluids take the simplified forms are immediately obtained from Eqs. (31)-(32) for $\lambda\to 0$.

$$u_s(y,t)=\frac{f}{\mu}\frac{e^{-yB}}{\sqrt{A^2+B^2}}sin\left(\omega t-yA+\varphi+\frac{\pi}{2}\right)-\frac{2f}{\pi\mu}\frac{\omega}{\nu}e^{\frac{-t}{2\lambda}}\int_0^{\infty}\frac{cos(y\xi)}{\left(\xi^2-\lambda\omega^2\nu^{-1}\right)^2+\omega^2\nu^{-2}}\times$$
$$\times\left\{cosh\left(\frac{c(\xi)}{2\lambda}t\right)+\frac{1-2\lambda\left(\nu\xi^2-\lambda\omega^2\right)}{c(\xi)}sinh\left(\frac{c(\xi)}{2\lambda}t\right)\right\}d\xi; \qquad (31)$$

$$\tau_s(y,t)=\frac{f}{\sqrt{1+(\lambda\omega)^2}}\left[e^{-yB}sin(\omega t-yA-\psi)+\frac{\lambda\omega e^{\frac{-t}{\lambda}}}{\sqrt{1+\lambda^2\omega^2}}\right]+\frac{2f}{\pi}\left(\frac{\omega}{\nu}\right)e^{\frac{-t}{2\lambda}}\int_0^\infty\frac{sin(y\xi)}{\xi\left[\left(\xi^2-\lambda\omega^2\nu^{-1}\right)^2+\omega^2\nu^{-2}\right]}\times$$

$$\times\left\{\left(\xi^2-\frac{\lambda\omega^2}{\nu}\right)cosh\left(\frac{c(\xi)}{2\lambda}t\right)+\frac{\left(\xi^2+\frac{\lambda\omega^2}{\nu}\right)}{c(\xi)}sinh\left(\frac{c(\xi)}{2\lambda}t\right)\right\}d\xi, \tag{32}$$

where $2A^2=\frac{\omega}{\nu}\left(\sqrt{1+\lambda^2\omega^2}+\lambda\omega\right)$, $2B^2=\frac{\omega}{\nu}\left(\sqrt{1+\lambda^2\omega^2}-\lambda\omega\right)$, $c(\xi)=\sqrt{1-4\lambda\nu\xi^2}$, $tan\varphi=\sqrt{1+\lambda^2\omega^2}-\lambda\omega.$

The similar solutions for Newtonian fluids [2], Eqs. (20) and (21):

$$u_{sN}(y,t)=\frac{f}{\mu}\sqrt{\frac{\nu}{\omega}}e^{-y\sqrt{\frac{\omega}{2\nu}}}sin\left(\omega t-y\sqrt{\frac{\omega}{2\nu}}+\frac{3\pi}{4}\right)-\frac{2f}{\mu\pi}\frac{\omega}{\nu}\int_0^\infty\frac{cos(y\xi)}{\xi^4+\omega^2\nu^{-2}}e^{-\nu\xi^2t}d\xi; \tag{33}$$

$$\tau_{sN}(y,t)=fe^{-y\sqrt{\frac{\omega}{2\nu}}}sin\left(\omega t-y\sqrt{\frac{\omega}{2\nu}}\right)+\frac{2f}{\pi}\frac{\omega}{\nu}\int_0^\infty\frac{\xi sin(y\xi)}{\xi^4+\omega^2\nu^{-2}}e^{-\nu\xi^2t}d\xi. \tag{34}$$

6. Numerical results, discussion and conclusions

In this note, oscillating motions of a Maxwell fluid between two side walls perpendicular to a plate are studied by means of integral transforms. The motion of the fluid is induced by plate that after time $t=0$ applies time-dependent shear stresses of the form Eq. (6) and Eq. (7) to the fluid. The obtained starting solutions for $u_s(y,z,t)$ and $\tau_{1s}(y,z,t)$, are presented as a sum between steady-state and transient solutions. They satisfy all imposed initial and boundary conditions and describe the motion of the fluid some time after its initiation. After that time when the transients disappear, the fluid is moving according to the steady state solutions. The similar solutions for Newtonian fluids, given by Eqs. (29) and (30), are recovered as limiting cases of general solutions.

In the absence of the side walls, namely when the distance between walls tends to infinity, the solutions that have been obtained reduce to the similar solutions Eqs. (31)-(34) corresponding to the motion over an infinite plate. In order to bring to light a new and useful application of present results, let us firstly remember a known solution [17], Eqs. (3.9) with $\alpha=0$:

$$\upsilon_{sN}(y,t)=Ve^{-y\sqrt{\frac{\omega}{2\nu}}}sin\left(\omega t-y\sqrt{\frac{\omega}{2\nu}}\right)+\frac{2V}{\pi}\frac{\omega}{\nu}\int_0^\infty\frac{\xi sin(y\xi)}{\xi^4+\left(\frac{\omega}{\nu}\right)^2}e^{-\nu\xi^2t}d\xi \tag{35}$$

for the velocity corresponding to the motion of a Newtonian fluid over an infinite plate that oscillates in its plane according to:

$$v(0,t)=V\,sin(\omega t). \tag{36}$$

As form, the right member of Eq. (35) is identical to those from Eq. (34) corresponding to the motion of a Newtonian fluid on an infinite plate that applies oscillating shear stresses $f\,sin(\omega t)$ to the fluid. This is not a surprise because for such motions of Newtonian fluids the velocity $\upsilon(y,t)$ and shear stress $\tau(y,t)$ satisfy the same governing equation, namely [12]:

$$\frac{\partial\upsilon_N(y,t)}{\partial t}=\mu\frac{\partial^2\upsilon_N(y,t)}{\partial y^2} \tag{37}$$

respectively, $\frac{\partial\tau_N(y,t)}{\partial t}=\mu\frac{\partial^2\tau_N(y,t)}{\partial y^2}$.

It is worth pointing out that such a property is also valid for Maxwell fluids. More exactly, the velocity $\upsilon_M(y,t)$ and the shear stress $\tau_M(y,t)$ corresponding to a Maxwell fluid in such a motion satisfy the governing equations

$$\left.\begin{aligned}\left(1+\lambda\frac{\partial}{\partial t}\right)\upsilon_M(y,t)&=\nu\frac{\partial^2\upsilon_M(y,t)}{\partial y^2};\\ \left(1+\lambda\frac{\partial}{\partial t}\right)\tau_M(y,t)&=\nu\frac{\partial^2\tau_M(y,t)}{\partial y^2}.\end{aligned}\right\} \tag{38}$$

This simple remark is very important from theoretical point of view. It allows us to provide new exact solutions for the motion of a Maxwell fluid over an infinite plate. For instance, the velocity of a Maxwell fluid over an infinite plate that after time $t=0^+$ oscillates in its plane according to:

$$v(0,t)=V\frac{\lambda\omega}{1+(\lambda\omega)^2}\left\{\frac{1}{\lambda\omega}sin(\omega t)-cos(\omega t)+e^{\frac{-t}{\lambda}}\right\} \tag{39}$$

is given by Eq. (32) with V instead of *f*.

Finally, in order to reveal some relevant physical aspects of the obtained results, some graphs are sketched in this section.

In order to determine the distance between the side walls for which the measured values of the velocity $u_s(y,z,t)$ in the middle of the channel are unaffected by the presence of the side walls, more exactly this velocity is equal to the velocity $u_s(y,t)$ corresponding to the motion over an infinite plate, Fig. 1 has been prepared. It is clearly seen from these diagrams that the distance between side walls has a significant influence on the velocity field. In the considered case, if $h > 0.55$ the influence of side walls on the velocity becomes insignificant. Other important problem regarding the technical relevance of starting solutions is to find the required time to reach the steady-state. More exactly, in practice it is necessary to know the approximate time after which the fluid is moving according to the steady-state solutions. To solve this problem, the variations of starting and steady-state velocities $u_s(y,t)$ and $u_{ss}(y,t)$ are depicted in Fig. 2. At small values of the time *t*, the difference between the starting and steady-state solutions is meaningful. This difference decreases in time and the required time to reach the steady-state for the motion due to the sine oscillations of the shear is decreasing if the frequency ω increases.

In conclusion, the main results obtained in this note are:

- Closed-form expressions for starting solutions corresponding to the oscillating motion of a Maxwell fluid between side walls perpendicular to a plate that applies oscillating shear stresses to the fluid are established in integral and series form.
- These solutions, that are presented as a sum of steady-state and transient solutions, describe the motion of the fluid some time after its initiation. After that time, when the transients disappear, the fluid is moving according to the steady-state solutions.
- Similar solutions corresponding to the motion over an infinite plate as well as the solutions for Newtonian fluids are obtained as limiting cases of general solutions.
- The distance between walls for which the velocity of the fluid in the middle of the channel is unaffected by their presence is graphically determined.
- Required time to reach the steady-state is a decreasing function with regard to the frequency ω of the shear stress.

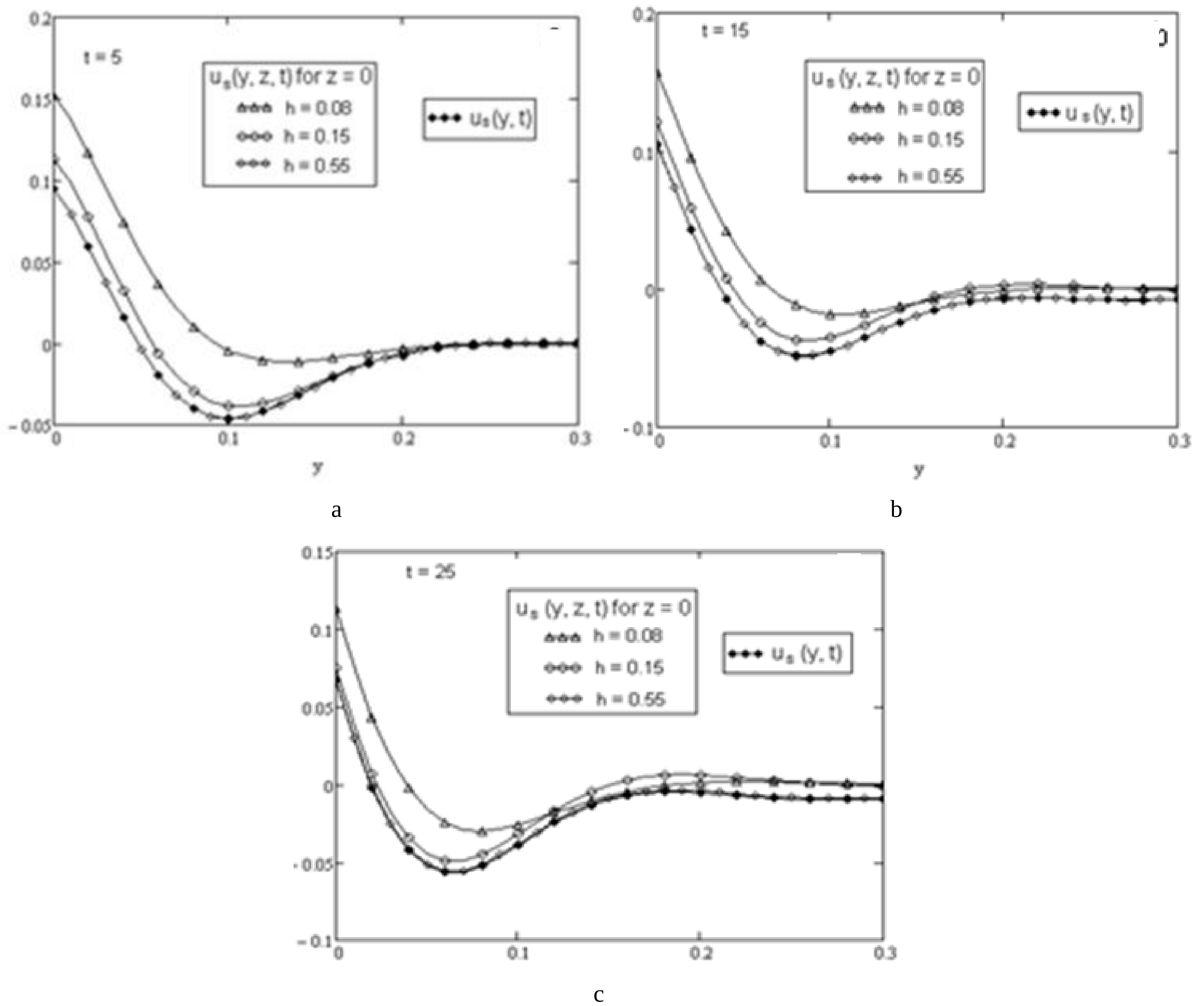
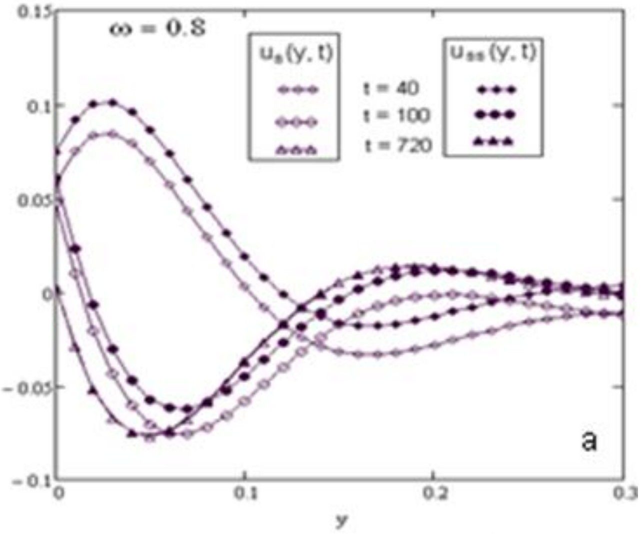
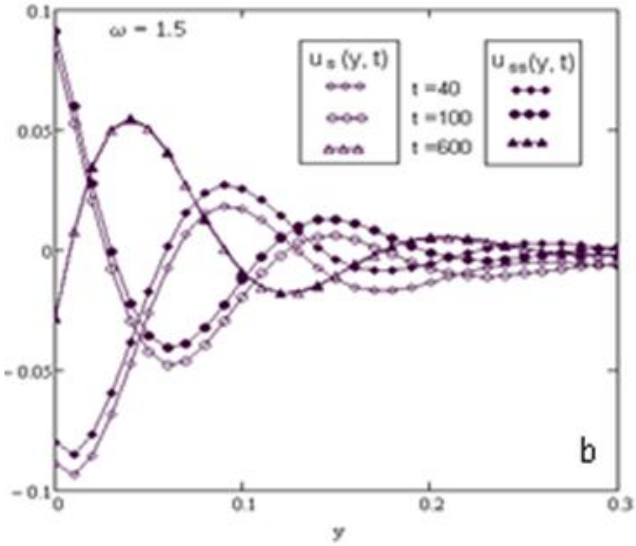


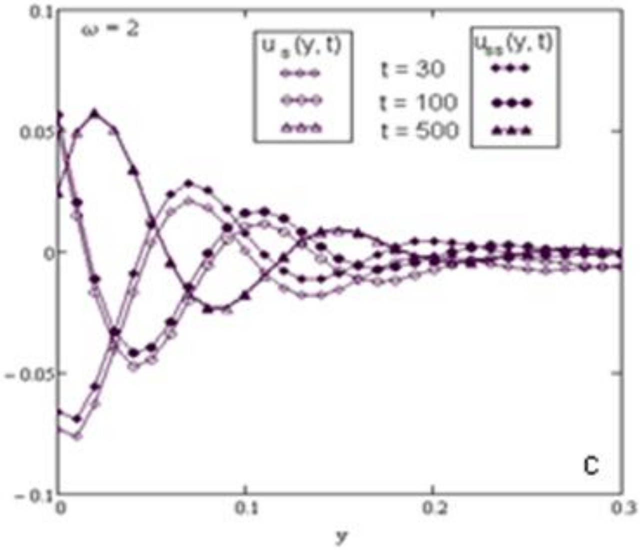
Fig. 1 Profiles of the velocities $u_s(y, 0, t)$ and $u_s(y, t)$ given by Eqs. (26) and (31)for $f = 6$ N/m^2, $\mu = 1.48$ Ns/m^2, $\nu = 0.001457$ m^2/s, $\lambda = 0.5$ s, $\omega = 1.2$ s^{-1} and different values of *t*, s and *h*, m



a



b



c

Fig. 2 The required time to reach the steady-state for sine oscillations of the shear .For $h \to \infty$, $\mu = 1.48$ Ns/m^2, $\nu = 0.001457$ m^2/s, $\lambda = 0.8$ s and different values of ω, s^{-1}

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ŠONINIŲ SIENELIŲ ĮTAKA MAKSVELO SKYSČIO PULSUOJAMAJAM JUDĖJIMUI VIRŠ BEGALINĖS PLOKŠTELĖS

R e z i u m ė

Pradinis vertinimas, atitinkantis Maksvelo skysčio pulsuojamąjį judėjimą tarp šoninių statmenų plokštelei sienelių, atliktas naudojant integrines transformacijas. Apie skysčio judėjimą dėl pulsuojančios šlyties poveikio sienelei literatūroje mažai kalbama. Sprendiniai, atitinkantys judėjimą virš begalinės plokštelės, kurią liečia pulsuojančio skysčio apkrova, yra gauti kaip ribiniai bendrojo sprendinio atvejai. Visi sprendiniai yra pateikti nusistovėjusio ir pereinamojo sprendinių sumos pavidalu ir lengvai gali būti konkretizuoti suteikiant panašius sprendinius niutoniniams skysčiams. Jie nusako skysčio judėjimą po kurio laiko nuo jo pradžios. Po šio laiko, kai pereinamasis procesas baigiasi, skysčio judėjimas išreiškiamas nusistovėjusiais sprendiniais, kurie yra periodiniai ir nepriklausomi nuo pradinių sąlygų. Tačiau jie tenkina ribines sąlygas ir vyraujančias lygtis. Grafiškai apibrėžti atstumas tarp sienelių, kuriam esant skysčio greičiui kanalo viduryje įtaka nedaroma, ir laikas būtinas, nusistovėjusiam režimui pasiekti.

A. Sohail, D. Vieru, M.A. Imran

INFLUENCE OF SIDE WALLS ON THE OSCILLATING MOTION OF A MAXWELL FLUID OVER AN INFINITE PLATE

S u m m a r y

Starting solutions corresponding to the oscillating motion of a Maxwell fluid between side walls perpendicular to a plate are established using integral transforms. Such solutions are scarce in the literature, the motion of the fluid being due to an oscillating shear on the boundary. The solutions corresponding to the motion over an infinite plate that applies an oscillating shear to the fluid are obtained as limiting cases of general solutions. All solutions are presented as a sum between steady-state and transient solutions and can easily be particularized to give the similar solutions for Newtonian fluids. They describe the motion of the fluid some time after its initiation. After that time, when the transients disappear, the motion of the fluid is described by the steady-state solutions which are periodic in time and independent of initial conditions. However, they satisfy the boundary conditions and governing equations. Finally, the distance between walls for which the velocity of the fluid in the middle of the channel in unaffected by their presence and the required time to reach the steady-state are graphically determined.

Keywords: Maxwell fluids, oscillating shear, side walls, starting solutions, steady-state and transient solutions.

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