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Stress-strain state of an elastic rectangular plate under dynamic load

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1. Introduction

The study of a high-speed deformation of structural materials is very important due to wide opportunities of its application in engineering. It should be accounted for in the assessment of the dynamic strength of structures under impact and shock loads either under technological operating conditions, or in emergency or critical situations. Detailed analysis of the high-speed deformation is also required in the derivation of processing parameters for formation and strengthening of structural elements with highly intensive energy sources. One of the distinctive features of these processes is dynamic properties of materials [1], which primarily concern theoretical foundations of materials science and, in particular, the study of the phenomenon of formation of dissipative structures under impact supply of deformation energy [2].

Although in general case the problem involves complicated, geometrically and physically nonlinear mechanical models [3], many important relations and effects that precede irreversible deformations can be observed and studied using linear models of elasticity [4].

This paper considers corresponding elastic problem for a plate with boundary conditions, which differ from traditional ones: instead of shear stress, corresponding components of displacement vector are equal to zero at surfaces, which are loaded with normal tension. This change in boundary conditions allowed agreeing the mathematical formulation of the problem with the experimental conditions during studies of the dynamic response of rectangular samples to impact loading [2] and to avoid traditional mathematical difficulties that arise while considering two-dimensional elastic problems for a rectangular domain [5].

2. Problem formulation and solution strategy

Consider a rectangular plate of a size $2h\times 2l$ at x_1 and y_1, respectively (Fig. 1). At the time $t=0$ the plate is loaded with a normal tension $p(t)$ applied to its fixed edges $x_1=\pm l$. Boundaries $y_1=\pm h$ are traction-free during the entire deformation process. The following dimensionless variables and constants are introduced: $x=x_1/l$, $y=y_1/l$, $x_0=h/l$, $\kappa^2=c_1/c_2=(\lambda+2\mu)/\mu$, $\tau=c_1t/l$ where c_1 and c_2 are the phase velocities of longitudinal and transverse waves for considered material of the plate, λ and μ are elastic moduli (Lame constants).

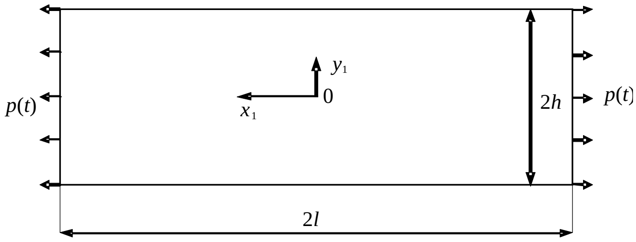

Fig. 1 Sketch of the problem

In terms of these variables, assuming that before the time $t=0$ the plate was at rest, the problem has the following mathematical formulation:

$$\frac{\partial^2\theta}{\partial x^2}+\frac{\partial^2\theta}{\partial y^2}=\frac{\partial^2\theta}{\partial \tau^2}; \tag{1}$$

$$\frac{\partial^2 u_y}{\partial x^2}+\frac{\partial^2 u_y}{\partial y^2}=\kappa^2\frac{\partial^2 u_y}{\partial \tau^2}-\left(\kappa^2-1\right)\frac{\partial\theta}{\partial y}; \tag{2}$$

$$\theta=\frac{\partial\theta}{\partial\tau}=0,\ u_y=\frac{\partial u_y}{\partial\tau}=0,\quad \tau=0; \tag{3}$$

$$\left.\begin{aligned}&u_y(\pm1,y,\tau)=0;\\&\sigma_{xx}(\pm1,y,\tau)=\pm p(\tau);\end{aligned}\right\} \tag{4}$$

$$\left.\begin{aligned}&\sigma_{xx}(x,\pm y_0,\tau)=0;\\&\sigma_{xy}(x,\pm y_0,\tau)=0,\end{aligned}\right\} \tag{5}$$

where $\theta(x,y,\tau)=\partial u_x/\partial x+\partial u_y/\partial y$ is the volumetric expansion, $u_x(x,y,\tau)$ and $u_y(x,y,\tau)$ are the components of elastic displacement vector, and

$$\left.\begin{aligned}&\sigma_{xx}=\lambda\theta+2\mu\varepsilon_{xx};\,\sigma_{yy}=\lambda\theta+2\mu\varepsilon_{yy};\,\sigma_{xy}=2\mu\varepsilon_{xy};\\&\varepsilon_{xx}=\frac{\partial u_x}{\partial x};\,\varepsilon_{yy}=\frac{\partial u_y}{\partial y};\varepsilon_{xy}=\frac{1}{2}\left(\frac{\partial u_x}{\partial y}+\frac{\partial u_y}{\partial x}\right)\end{aligned}\right\} \tag{6}$$

are the components of stress and strain tensors, respectively.

From the conditions (4), accounting for:

$$\theta(\pm1,y,\tau)\equiv\left.\left(\frac{\partial u_x}{\partial x}+\frac{\partial u_y}{\partial y}\right)\right|_{x=\pm1}=\left.\frac{\partial u_x}{\partial x}\right|_{x=\pm1},$$

it follows that

$$\mu^{-1}\sigma_{xx}(\pm1,y,\tau)=\kappa^2\theta(\pm1,y,\tau). \tag{7}$$

The Laplace integral transform for a time variable [6] and a finite Fourier cosine transform for a spatial variable x [7] can be applied to Eq. (1). Accounting for symmetry of the problem, zero initial conditions (3), relations (7) and conditions (4), Eq. (1) writes as:

$$\frac{d^2\bar{\theta}_n}{dy^2}-\left(\xi_n^2+s^2\right)\bar{\theta}_n=(-1)^{n+1}\frac{2\xi_n}{\kappa\mu}\bar{p}(s), \tag{8}$$

where $\xi_n=\pi(2n+1)/2,\ n=0,1,2,\ldots,$ and $\bar{\theta}_n(y,s)=\int_{-1}^{1}cos(\xi_n x)\int_0^\infty\theta(x,y,\tau)exp(-s\tau)d\tau dx$ is Laplace and Fourier dual transform.

Applying the same dual transform to Eq. (2) one obtains:

$$\frac{d^2\bar{v}_n}{dy^2}-\left(\xi_n^2+\kappa^2s^2\right)\bar{v}_n=\left(1-\kappa^2\right)\frac{d\bar{\theta}_n}{dy}, \tag{9}$$

where $\bar{v}_n(y,s)=\int_{-1}^{1}cos(\xi_n x)\int_0^\infty u_y(x,y,\tau)exp(-s\tau)d\tau dx$.

Accounting for the fact, that the function $\bar{\theta}_n(y,s)$ is even with respect to the argument y, the solution of Eq. (7) is as follows:

$$\bar{\theta}_n(y,s)=A_n(s)cosh(\gamma_1 y)+\frac{(-1)^n 2\xi_n\bar{p}(s)}{\mu\kappa^2\gamma_1^2}, \tag{10}$$

with $\gamma_1=\sqrt{\xi_n^2+s^2}$.

Accounting for (10), the solution of Eq. (9) writes as:

$$\bar{v}_n(y,s)=B_n(s)sinh(\gamma_2 y)+\frac{\gamma_1}{s^2}A_n(s)sinh(\gamma_1 y) \tag{11}$$

with $\gamma_2=\sqrt{\xi_n^2+\kappa^2s^2}$.

Another component $\bar{u}_n(y,s)=\int_{-1}^{1}sin(\xi_n x)\int_0^\infty u_x(x,y,\tau)exp(-s\tau)d\tau dx$ of the displacement vector can be obtained with the account of the relation $\bar{u}_n=\frac{1}{\xi_n}\left(\bar{\theta}_n-\frac{d\bar{v}_n}{dy}\right)$ as:

$$\begin{aligned}\bar{u}_n(y,s)=&-B_n(s)\xi_n^{-1}\gamma_2 cosh(\gamma_2 y)-\\&-\frac{\xi_n}{s^2}A_n(s)cosh(\gamma_1 y)+\frac{(-1)^n 2\bar{p}(s)}{\mu\kappa^2\gamma_1^2}.\end{aligned} \tag{12}$$

Two quantities $A_n(s)$ and $B_n(s)$ can be obtained based on the boundary conditions (5), which dual Laplace and Fourier transform writes as

$$\left.\begin{aligned}&\left(\kappa^2-2\right)\bar{\theta}_n+2\frac{d\bar{u}_n}{dy}=0,\ y=\pm y_0;\\&-\xi_n\bar{v}_n+\frac{d\bar{u}_n}{dy}=0,\ y=\pm y_0.\end{aligned}\right\} \tag{13}$$

Accounting for Eqs. (10)-(12), from Eq. (13) one obtains

$$\left.\begin{aligned}&A_n(s)=\frac{s^2\left(\xi_n^2+\gamma_2^2\right)sinh(\gamma_2 y_0)\bar{p}_n(s)}{\gamma_1^2\Delta(\xi,s)};\\&B_n(s)=-2\xi_n^2\,sinh(\gamma_1 y_0)\bar{p}_n(s)\big/\left[\gamma_1\Delta(\xi,s)\right],\end{aligned}\right\} \tag{14}$$

where

$$\Delta(\xi,s)=4\xi_n^2\gamma_1\gamma_2\,sinh(\gamma_1 y_0)cosh(\gamma_2 y_0)-\left(\xi_n^2+\gamma_2^2\right)^2\times$$
$$\times cosh(\gamma_1 y_0)sinh(\gamma_2 y_0),\ \bar{p}_n(s)=\frac{(-1)^n 2\xi_n\left(\kappa^2-2\right)\bar{p}(s)}{\mu\kappa^2}.$$

Finally, accounting for obtained values of $A_n(s)$ and $B_n(s)$ the transforms of transient displacements are equal to

$$\left.\begin{aligned}&\bar{v}_n(y,s)=\gamma_1^{-1}\Delta^{-1}\left[\left(\xi_n^2+\gamma_2^2\right)sinh(\gamma_2 y_0)sinh(\gamma_1 y)-\right.\\&\qquad\left.-2\xi_n^2\,sinh(\gamma_1 y_0)sinh(\gamma_2 y)\right]\bar{p}_n;\\&\bar{u}_n(y,s)=\left(\frac{1}{\left(\kappa^2-2\right)\xi_n}+\frac{\xi_n}{\Delta}\left[2\gamma_1\gamma_2\,sinh(\gamma_1 y_0)\times\right.\right.\\&\left.\left.\times cosh(\gamma_2 y)-\left(\xi_n^2+\gamma_2^2\right)sinh(\gamma_2 y_0)cosh(\gamma_1 y)\right]\right)\frac{\bar{p}_n}{\gamma_1^2}.\end{aligned}\right\} \tag{15}$$

Inverse Laplace transform is obtained using the partial fraction expansions theorem [6, 8]. For this purpose one should first consider the first expression in Eq. (15) and solve it for singular points of the denominator. It is obvious that the roots of the equation $\gamma_1=0$ are not the singular points of the denominator, therefore consider the following equation:

$$\begin{aligned}&4\xi_n^2\gamma_1\gamma_2\,sinh(\gamma_1 y_0)cosh(\gamma_2 y_0)-\\&-\left(\xi_n^2+\gamma_2^2\right)^2cosh(\gamma_1 y_0)sinh(\gamma_2 y_0)=0.\end{aligned} \tag{16}$$

The roots of the characteristic Eq. (16) are purely imaginary and complex conjugate. Therefore, it is convenient to change variables with $s=i\eta$ to obtain $\gamma_1=\sqrt{\xi_n^2-\eta^2}$ and $\gamma_2=\sqrt{\xi_n^2-\kappa^2\eta^2}$, respectively. It is obvious that the roots $\eta_{n,k}$ depend on the discrete parameter ξ_n, therefore, there are three possible cases of there location:

$$0\le\left|\eta_{n,k}\right|\le\kappa^{-1}\xi_n;\ \kappa^{-1}\xi_n<\left|\eta_{n,k}\right|\le\xi_n;\ \left|\eta_{n,k}\right|>\xi_n. \tag{17}$$

For the first interval, the characteristic equation preserves its form (16) and has finite number of the roots $\eta_{n,k,1}$.

As for the interval $\kappa^{-1}\xi_n<\left|\eta_{n,k}\right|\le\xi_n$, the characteristic equation writes as:

$$\begin{aligned}&4\xi_n^2\gamma_1\tilde{\gamma}_2\,sinh(\gamma_1 y_0)cos(\tilde{\gamma}_2 y_0)-\\&-\left(\xi_n^2+\tilde{\gamma}_2^2\right)^2cosh(\gamma_1 y_0)sin(\tilde{\gamma}_2 y_0)=0,\end{aligned} \tag{18}$$

for $\tilde{\gamma}_2 = \sqrt{\kappa^2\eta^2 - \xi_n^2}$. Eq. (18) has a finite number k_2 of the roots $\eta_{n,k,2}$.

And, respectively, for the interval $|\eta_{n,k}| > \xi_n$ the equation

$$4\xi_n^2\tilde{\gamma}_1\tilde{\gamma}_2\, sin(\tilde{\gamma}_1 y_0)cos(\tilde{\gamma}_2 y_0) + \\ +(\xi_n^2 + \tilde{\gamma}_2^2)^2 cos(\tilde{\gamma}_1 y_0)sin(\tilde{\gamma}_2 y_0) = 0, \qquad (19)$$

for $\tilde{\gamma}_1 = \sqrt{\eta^2 - \xi_n^2}$ has an infinite number of the roots $\eta_{n,k,3}$.

To obtain the inverse of Eq. (15) using the partial fraction expansions theorem, one should evaluate the derivative of the denominator [6, 8]. To apply this, consider a derivative of the expression for $\Delta(\xi, s)$ for different intervals of location of the roots of the characteristic equation:

$$0 < |\eta| \le \frac{\xi_n}{\kappa}; \quad \left.\frac{d\Delta}{ds}\right|_{s=\pm\eta_{n,k,1}} \equiv \tilde{\Delta}_1(n,k) = \pm i\eta_{n,k,1}\left\{4\xi_n^2\left[\frac{\gamma_2}{\gamma_1}sinh(\gamma_1 y_0)cosh(\gamma_2 y_0) + \frac{\kappa^2\gamma_1}{\gamma_2}sinh(\gamma_1 y_0)cosh(\gamma_2 y_0) + \right.\right.$$
$$\left. + x_0\gamma_2\, cosh(\gamma_1 y_0)cosh(\gamma_2 y_0) + \kappa^2 y_0\gamma_1\, sinh(\gamma_1 y_0)sinh(\gamma_2 y_0)\right] - 4\kappa^2(\xi_n^2 + \gamma_2^2)cosh(\gamma_1 y_0)sinh(\gamma_2 y_0) -$$
$$\left. -(\xi_n^2 + \gamma_2^2)^2\, y_0\, sinh(\gamma_1 y_0)sinh(\gamma_2 y_0)/\gamma_1 - (\xi_n^2 + \gamma_2^2)^2\frac{\kappa^2 y_0}{\gamma_2}cosh(\gamma_1 y_0)cosh(\gamma_2 y_0)\right\}; \qquad (20)$$

$$\frac{\xi_n}{\kappa} < |\eta| \le \xi_n; \quad \left.\frac{d\Delta}{ds}\right|_{s=\pm\eta_{n,k,2}} \equiv \tilde{\Delta}_2(n,k) = \mp\eta_{n,k,2}\left\{4\xi_n^2\left[\frac{\tilde{\gamma}_2}{\gamma_1}sinh(\gamma_1 y_0)cos(\tilde{\gamma}_2 y_0) - \frac{\kappa^2\gamma_1}{\tilde{\gamma}_2}sinh(\gamma_1 y_0)cos(\tilde{\gamma}_2 y_0) + \right.\right.$$
$$\left. + x_0\tilde{\gamma}_2\, cosh(\gamma_1 y_0)cos(\tilde{\gamma}_2 y_0) + \kappa^2 y_0\gamma_1\, sinh(\gamma_1 y_0)sin(\tilde{\gamma}_2 y_0)\right] - 4\kappa^2(\xi_n^2 - \tilde{\gamma}_2^2)cosh(\gamma_1 y_0)sin(\tilde{\gamma}_2 y_0) -$$
$$\left. -(\xi_n^2 - \tilde{\gamma}_2^2)^2\, y_0\, sinh(\gamma_1 y_0)sin(\tilde{\gamma}_2 y_0)/\gamma_1 + (\xi_n^2 - \tilde{\gamma}_2^2)^2\frac{\kappa^2 y_0}{\tilde{\gamma}_2}cosh(\gamma_1 y_0)cos(\tilde{\gamma}_2 y_0)\right\}; \qquad (21)$$

$$|\eta| > \xi_n; \quad \left.\frac{d\Delta}{ds}\right|_{s=\pm\eta_{n,k,3}} \equiv \tilde{\Delta}_3(n,k) = \mp\eta_{n,k,3}\left\{4\xi_n^2\left[\frac{\tilde{\gamma}_2}{\tilde{\gamma}_1}sin(\tilde{\gamma}_1 y_0)cos(\tilde{\gamma}_2 y_0) + \frac{\kappa^2\tilde{\gamma}_1}{\tilde{\gamma}_2}sin(\tilde{\gamma}_1 y_0)cos(\tilde{\gamma}_2 y_0) + \right.\right.$$
$$\left. + y_0\tilde{\gamma}_2\, cos(\tilde{\gamma}_1 y_0)cos(\tilde{\gamma}_2 y_0) - \kappa^2 y_0\tilde{\gamma}_1\, sin(\tilde{\gamma}_1 y_0)sin(\tilde{\gamma}_2 y_0)\right] - 4\kappa^2(\xi_n^2 - \tilde{\gamma}_2^2)cos(\tilde{\gamma}_1 y_0)sin(\tilde{\gamma}_2 y_0) -$$
$$\left. -(\xi_n^2 - \tilde{\gamma}_2^2)^2\frac{y_0}{\tilde{\gamma}_1}sin(\tilde{\gamma}_1 y_0)sin(\tilde{\gamma}_2 y_0) + (\xi_n^2 - \tilde{\gamma}_2^2)^2\frac{\kappa^2 y_0}{\tilde{\gamma}_2}cos(\tilde{\gamma}_1 y_0)cos(\tilde{\gamma}_2 y_0)\right\}. \qquad (22)$$

The second expression in Eq. (15) besides the roots of the characteristic equation (16) has singular points placed at the roots of the equation $\gamma_1 = 0$: $s = \pm i\xi$. Accounting for this, the final expressions for transient components of the displacement vector write as:

$$\left.\begin{array}{l}
u_x(x,y,\tau) = \dfrac{4}{\mu}\left(1-\dfrac{2}{\kappa^2}\right)\displaystyle\sum_{n=0}^{\infty}(-1)^n\,\xi_n^2\left\{\sum_{k=1}^{k_1}\frac{2\gamma_1\gamma_2\, sinh(\gamma_1 y_0)cosh(\gamma_2 y) - (2\xi_n^2 - \kappa^2\eta_{n,k,1}^2)sinh(\gamma_2 y_0)cosh(\gamma_1 y)}{\gamma_1^2\tilde{\Delta}_1(n,k)}\times\right.\\
\quad\times f(\eta_{n,k,1},\tau) + \displaystyle\sum_{k=1}^{k_2}\frac{2\gamma_1\tilde{\gamma}_2\, sinh(\gamma_1 y_0)cos(\tilde{\gamma}_2 y) - (2\xi_n^2 - \kappa^2\eta_{n,k,2}^2)sin(\tilde{\gamma}_2 y_0)cosh(\gamma_1 y)}{\gamma_1^2\tilde{\Delta}_2(n,k)}f(\eta_{n,k,2},\tau) +\\
\quad\left.+\displaystyle\sum_{k=1}^{\infty}\frac{2\tilde{\gamma}_1\tilde{\gamma}_2\, sin(\tilde{\gamma}_1 y_0)cos(\tilde{\gamma}_2 y) + (2\xi_n^2 - \kappa^2\eta_{n,k,3}^2)sin(\tilde{\gamma}_2 y_0)cos(\tilde{\gamma}_1 y)}{\tilde{\gamma}_1^2\tilde{\Delta}_3(n,k)}f(\eta_{n,k,3},\tau)\right\}sin(\xi_n x);\\
u_y(x,y,\tau) = \dfrac{4}{\mu}\left(1-\dfrac{2}{\kappa^2}\right)\displaystyle\sum_{n=0}^{\infty}(-1)^n\,\xi_n\left\{\sum_{k=1}^{k_1}\frac{(2\xi_n^2 - \kappa^2\eta_{n,k,1}^2)sinh(\gamma_2 y_0)sinh(\gamma_1 y) - 2\xi_n^2\, sinh(\gamma_1 y_0)sinh(\gamma_2 y)}{\gamma_1\tilde{\Delta}_1(n,k)}\times\right.\\
\times f(\eta_{n,k,1},\tau) + \displaystyle\sum_{k=1}^{k_2}\frac{(2\xi_n^2 - \kappa^2\eta_{n,k,2}^2)sin(\tilde{\gamma}_2 y_0)sinh(\gamma_1 y) - 2\xi_n^2\, sinh(\gamma_1 y_0)sin(\tilde{\gamma}_2 y)}{\gamma_1\tilde{\Delta}_2(n,k)}f(\eta_{n,k,2},\tau) +\\
\quad\left.+\displaystyle\sum_{k=1}^{\infty}\frac{(2\xi_n^2 - \kappa^2\eta_{n,k,3}^2)sin(\tilde{\gamma}_2 y_0)sin(\tilde{\gamma}_1 y) - 2\xi_n^2\, sin(\tilde{\gamma}_1 y_0)sin(\tilde{\gamma}_2 y))}{\tilde{\gamma}_1\tilde{\Delta}_3(n,k)}f(\eta_{n,k,3},\tau)\right\}cos(\xi_n x),
\end{array}\right\} \qquad (23)$$

where

$$f(\eta,\tau)=\int_0^{\tau} p(\tau-t)\sin(\eta t)\,dt. \tag{24}$$

Eq. (23) expresses exact closed-form solution of the transient dynamic problem of elasticity for a rectangular plate under arbitrarily time-dependent external load.

The components of strain and stress tensors are evaluated by applying strain-displacement and stress-strain relations (6) to the obtained displacement components (23). Also it can be shown that all series in the solution (23) converges uniformly, therefore differentiation operators used in derivation of stress and strain tensors can be applied directly under the sum sign.

Particularly, to evaluate stresses at the arbitrary point of the plate based on Eqs. (6) and (23) one obtains the following formulae

$$\begin{aligned}\sigma_{xx}(x,y,\tau)&=4\left(1-2/\kappa^2\right)\sum_{n=0}^{\infty}(-1)^n\,\xi_n\times\\&\times\left\{\sum_{k=1}^{k_1}\frac{2\gamma_1\gamma_2\sinh(\gamma_1 y_0)\cosh(\gamma_2 y)-\left(2\xi_n^2-\kappa^2\eta_{n,k,1}^2\right)\left(2\xi_n^2+\left(\kappa^2-2\right)\eta_{n,k,1}^2\right)\sinh(\gamma_2 y_0)\cosh(\gamma_1 y)}{\gamma_1^2\tilde{\Delta}_1(n,k)}f(\eta_{n,k,1},\tau)+\right.\\&+\sum_{k=1}^{k_2}\frac{2\gamma_1\tilde{\gamma}_2\sinh(\gamma_1 y_0)\cos(\tilde{\gamma}_2 y)-\left(2\xi_n^2-\kappa^2\eta_{n,k,2}^2\right)\left(2\xi_n^2+\left(\kappa^2-2\right)\eta_{n,k,2}^2\right)\sin(\tilde{\gamma}_2 y_0)\cosh(\gamma_1 y)}{\gamma_1^2\tilde{\Delta}_2(n,k)}f(\eta_{n,k,2},\tau)+\\&\left.+\sum_{k=1}^{\infty}\frac{2\tilde{\gamma}_1\tilde{\gamma}_2\sin(\tilde{\gamma}_1 y_0)\cos(\tilde{\gamma}_2 y)+\left(2\xi_n^2-\kappa^2\eta_{n,k,3}^2\right)\left(2\xi_n^2+\left(\kappa^2-2\right)\eta_{n,k,3}^2\right)\sin(\tilde{\gamma}_2 y_0)\cos(\tilde{\gamma}_1 y)}{\tilde{\gamma}_1^2\tilde{\Delta}_3(n,k)}f(\eta_{n,k,3},\tau)\right\}\cos(\xi_n x);\end{aligned}\tag{25}$$

$$\begin{aligned}\sigma_{yy}(x,y,\tau)&=4\left(1-2/\kappa^2\right)\sum_{n=0}^{\infty}(-1)^{n+1}\,\xi_n\left\{\sum_{k=1}^{k_1}\frac{2\gamma_1\gamma_2\sinh(\gamma_1 y_0)\cosh(\gamma_2 y)-\left(2\xi_n^2-\kappa^2\eta_{n,k,1}^2\right)^2\sinh(\gamma_2 y_0)\cosh(\gamma_1 y)}{\gamma_1^2\tilde{\Delta}_1(n,k)}\times\right.\\&\times f(\eta_{n,k,1},\tau)+\sum_{k=1}^{k_2}\frac{2\gamma_1\tilde{\gamma}_2\sinh(\gamma_1 y_0)\cos(\tilde{\gamma}_2 y)-\left(2\xi_n^2-\kappa^2\eta_{n,k,2}^2\right)^2\sin(\tilde{\gamma}_2 y_0)\cosh(\gamma_1 y)}{\gamma_1^2\tilde{\Delta}_2(n,k)}f(\eta_{n,k,2},\tau)+\\&\left.+\sum_{k=1}^{\infty}\frac{2\tilde{\gamma}_1\tilde{\gamma}_2\sin(\tilde{\gamma}_1 y_0)\cos(\tilde{\gamma}_2 y)+\left(2\xi_n^2-\kappa^2\eta_{n,k,3}^2\right)^2\sin(\tilde{\gamma}_2 y_0)\cos(\tilde{\gamma}_1 y)}{\tilde{\gamma}_1^2\tilde{\Delta}_3(n,k)}f(\eta_{n,k,3},\tau)\right\}\cos(\xi_n x);\end{aligned}\tag{26}$$

$$\begin{aligned}\sigma_{xy}(x,y,t)&=4\left(1-2/\kappa^2\right)\sum_{n=0}^{\infty}(-1)^n\,\xi_n^2\left\{\sum_{k=1}^{k_1}\frac{\sinh(\gamma_1 x_0)\sinh(\gamma_2 x)-\sinh(\gamma_2 x_0)\sinh(\gamma_1 x)}{\gamma_1\tilde{\Delta}_1(n,k)}\left(2\xi_n^2-\kappa^2\eta_{n,k,1}^2\right)\times\right.\\&\times f(\eta_{n,k,1},\tau)+\sum_{k=1}^{k_2}\frac{\sinh(\gamma_1 x_0)\sin(\tilde{\gamma}_2 x)-\sin(\tilde{\gamma}_2 x_0)\sinh(\gamma_1 x)}{\gamma_1\tilde{\Delta}_2(n,k)}\left(2\xi_n^2-\kappa^2\eta_{n,k,2}^2\right)f(\eta_{n,k,2},\tau)+\\&\left.+\sum_{k=1}^{\infty}\frac{\sin(\tilde{\gamma}_1 x_0)\sin(\tilde{\gamma}_2 x)-\sin(\tilde{\gamma}_2 x_0)\sin(\tilde{\gamma}_1 x)}{\tilde{\gamma}_1\tilde{\Delta}_3(n,k)}\left(2\xi_n^2-\kappa^2\eta_{n,k,3}^2\right)f(\eta_{n,k,3},\tau)\right\}\sin(\xi_n y).\end{aligned}\tag{27}$$

3. Numerical results

In practice, dynamic impact loading is always continuous in time, and increase faster or slower from zero value to the limiting one. Therefore, the high-speed increase in load is approximated with a relation $p(t)=p^*\left(1-exp(-at)\right)^2$, which can be expressed in terms of the dimensionless time τ as follows

$$p(\tau)=p^*\left(1-exp(-\tau_0\tau)\right)^2, \tag{28}$$

where $\tau_0=(l\times a)/c_1$ is a loading rate, p^* is a dimensional parameter (Pa). This relation allows agreeing initial and boundary conditions, and also it allows accurate approximation of the real dependence of dynamic load on time in many practically important cases (Fig. 2).

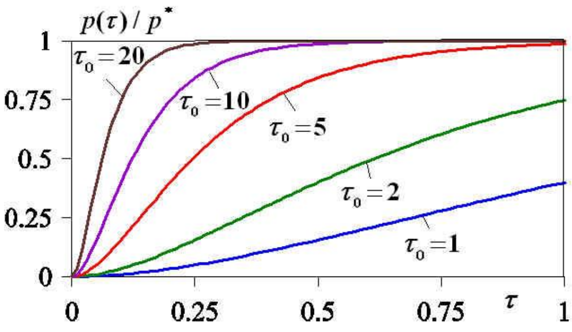

Fig. 2 Dependence of the applied load on the dimensionless time

Numerical calculations are held for a plate made of aluminum alloy 2024-T3 [9], which Young modulus and Poisson ration are equal to $E=6.9$ GPa, $\nu=0.3$, respectively. One should note that the value of the dimensionless time $\tau = 1$ correspond to the time $t = 0.02$ ms for the length of the plate, which equals $2l = 0.3$ m .

Evaluating the integral (24) for the load-time dependence given by Eq. (28) one obtains explicit expression for the function $f(\eta, \tau)$:

$$f(\eta,\tau) = p^* \left\{ \frac{1}{\eta} - \frac{2\eta\, exp(-\tau_0\tau)}{\eta^2+\tau_0^2} + \frac{\eta\, exp(-2\tau_0\tau)}{\eta^2+4\tau_0^2} + \right.$$
$$+ cos(\eta\tau)\left(\frac{2\eta}{\eta^2+\tau_0^2} - \frac{\eta}{\eta^2+4\tau_0^2} - \frac{1}{\eta} \right) +$$
$$\left. + sin(\eta\tau)\left(\frac{2\tau_0}{\eta^2+4\tau_0^2} - \frac{2\tau_0}{\eta^2+\tau_0^2} \right) \right\}. \qquad (29)$$

Fig. 3 shows the time dependence of the dimensionless longitudinal stress σ_{xx} / p^* at the point, which is located at the symmetry axis of the plate ($x = 0.5,\ y = 0$). The stress is calculated by means of Eqs. (25) and (29) for different values of the parameter τ_0 (loading rate).

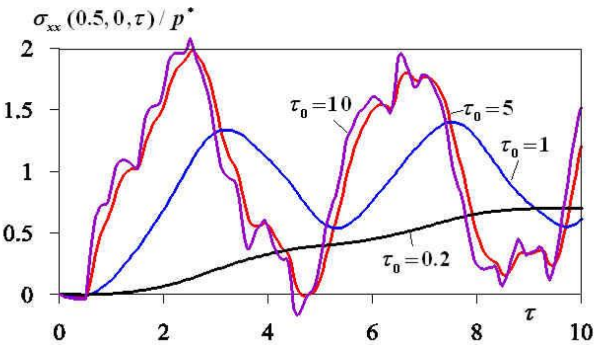

Fig. 3 Longitudinal stress in the plate obtained for different values of the loading rate

One can see in Fig. 3 that the time dependence of the longitudinal stress essentially depends on the parameter τ_0: for $\tau_0 = 0.2$ this dependence is quasi-static one (repeats time dependence of the external load), and for $\tau_0 > 10$ it practically coincides with vibrations caused by the abrupt (shock) loading $p(\tau) = p^* S_+(\tau)$. Increase in the parameter τ_0 also causes increase in the amplitude of vibrations and decrease in time of the arrival of the fist wave-front.

The same conclusions can be made for dynamic distributions of the dimensionless transverse stress at the same point (Fig. 4) and the dimensionless shear stress at the point $x = 0.5,\ y = 0.15$ (Fig. 5) depending on the parameter τ_0.

However, unlike previous results, the influence of the parameter τ_0 on the amplitude of vibrations is more significant: the increase in the loading rate parameter from the value of $\tau_0 = 1$ to $\tau_0 = 5$ causes increase in the amplitude of transverse stress up to three times, and increases the shear stress up to two times. At the same time the longitudinal stress increases at only 40–45%.

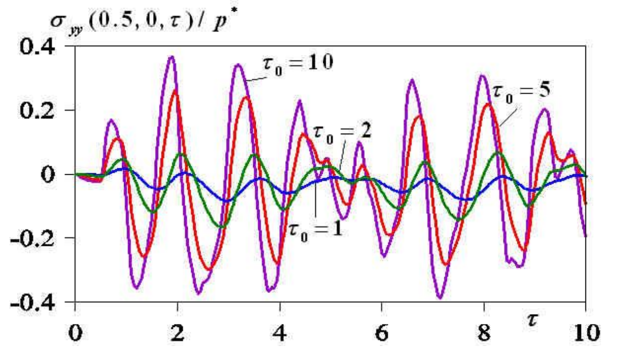

Fig. 4 Transverse stress in the plate obtained for different values of the loading rate

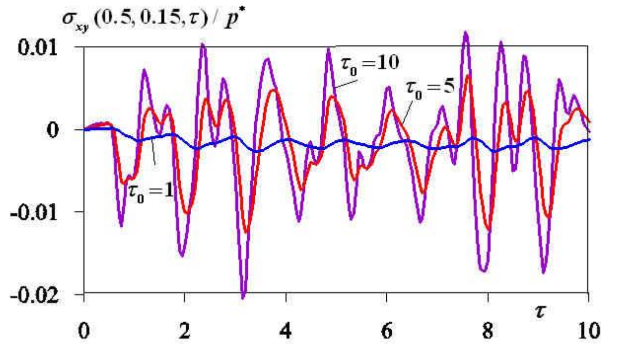

Fig. 5 Shear stress in the plate obtained for different values of the loading rate

Besides, one can see that maximal magnitude of the longitudinal stress is much higher than corresponding of transverse and shear stresses. In this context, one can arise a question about the difference between the obtained distributions of longitudinal stress and the corresponding one, which is obtained by neglecting the change of elastic field parameters in the *y* direction (one-dimensional model).

In the case of the one-dimensional model, based on relations (28), the longitudinal stress in the plate are defined by the following formula:

$$\frac{\sigma_{xx}(x,\tau)}{p^*} = 2\sum_{n=0}^{\infty}(-1)^n \left[\frac{1}{\xi_n} - \frac{2\xi_n\, exp(-\tau_0\tau)}{\xi_n^{\,2}+\tau_0^2} + \right.$$
$$+ \frac{\xi_n\, exp(-2\tau_0\tau)}{\xi_n^{\,2}+4\tau_0^2} + cos(\xi_n\tau) \times$$
$$\times \left(\frac{2\xi_n}{\xi_n^{\,2}+\tau_0^2} - \frac{\xi_n}{\xi_n^{\,2}+4\tau_0^2} - \frac{1}{\xi_n} \right) +$$
$$\left. + sin(\xi_n\tau)\left(\frac{2\tau_0}{\xi_n^{\,2}+4\tau_0^2} - \frac{2\tau_0}{\xi_n^{\,2}+\tau_0^2} \right) \right] cos(\xi_n x). \qquad (30)$$

Fig. 6 presents the comparative analysis of distributions of the dimensionless longitudinal stress inside the plate, obtained based on Eqs. (25) and (30) for $\tau_0 = 5$.

One can see in Fig. 6 that the error of calculations obtained based on the one-dimensional model does not exceed 10-15% comparing to the two-dimensional one. Accounting for the abovementioned results for shear and transverse stresses, this allows using a simple one-dimensional model in the applied engineering analysis.

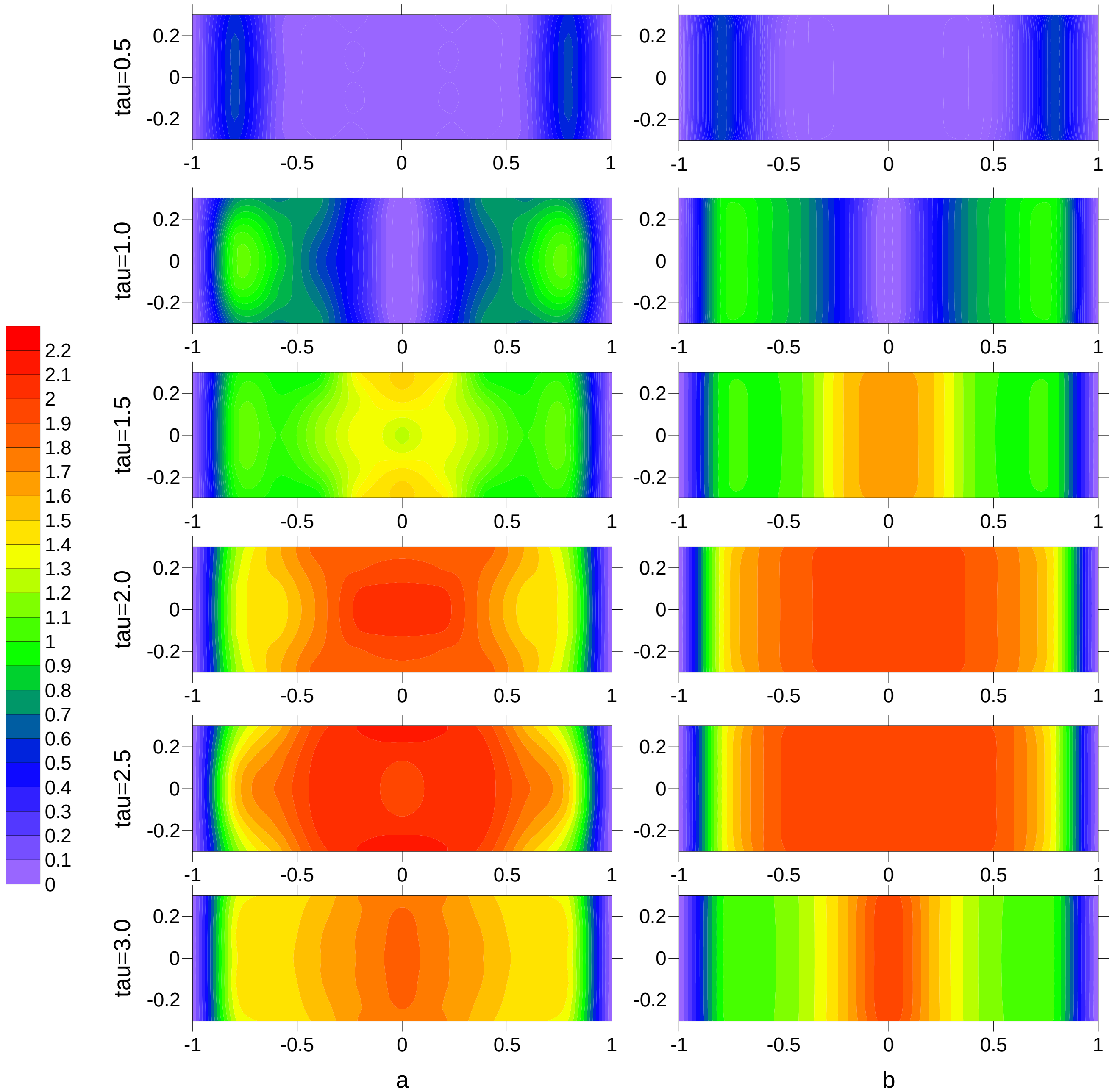

Fig. 6 Distribution of longitudinal stress in the plate for different moments of time obtained based on the two-dimensional (a) and one-dimensional (b) models

4. Conclusions

Using the Laplace transform for the time variable this paper solves the transient dynamic problem of elasticity for a rectangular plate in the form of double trigonometric series. The solution, which models the high-speed elastic deformation of materials, is obtained with the assumption that the surfaces, which are loaded with normal tension, are constrained such that during the transient deformation tangential stresses are equal to zero at these surfaces. This assumption allowed bypassing traditional mathematical problem, which is raised in the study of transient dynamic problems for finite two-dimensional domains, and allowed further application of Fourier integral transform for the spatial variable.

Based on the obtained analytic solution the numerical analysis of the transient stress dependence on the loading rate is held for a rectangular plate made of the aluminum alloy 2024-T3. It is observed that the increase in the loading rate essentially increases the magnitude of vibrations, and only slightly influence their frequency. Vibrations have harmonic behavior, since the mathematical model used does not consider damping.

Based on the comparative analysis of the obtained solution with the corresponding solution of the one-dimensional problem it is shown that for the considered load and boundary conditions the difference between the results of both approaches is insignificant, therefore, one can use a simpler one-dimensional model in the applied engineering analysis.

Acknowledgments

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TAMPRIOS STAČIAKAMPĖS PLOKŠTELĖS ĮTEMPIŲ IR DEFORMACIJŲ BŪVIS ESANT DINAMINĖMS APKROVOMS

R e z i u m ė

Darbe, siekiant modeliuoti eksperimentinių pavyzdžių tamprųjį įtempių ir deformacijų būvį, esant greitaveikei apkrovai, išspręstas stačiakampės plokštelės, apkrautos laikui bėgant kintančia tempimo apkrova, tamprumo teorijos uždavinys. Taikant integralinių pakeitimų uždavinio sprendimo metodiką, ištirta apkrovos greičio įtaka pereinamiesiems aliuminio lydinio plokštelės įtempiams. Aptariama galimybė supaprastintą vienmatį modelį panaudoti inžineriniuose tyrimuose.

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STRESS-STRAIN STATE OF AN ELASTIC RECTANGULAR PLATE UNDER DYNAMIC LOAD

S u m m a r y

This paper considers a dynamic problem of elasticity for a rectangular plate under time-dependent tensile load, and uses it in modeling of transient stress-strain state of experimental samples under high-speed dynamic overloading.

The solution of the problem is obtained using the integral transform approach. Based on this solution the paper studies the influence of loading rate on the transient stress in plate made of the aluminium alloy. It discusses the possibility of application in engineering analysis of a simpler one-dimensional model, which is also adopted.

Keywords: dynamic problem of elasticity, integral transform, loading rate.

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