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The application of 2-dimensional elasticity for the elastic analysis of solid sphere made of exponential functionally graded material

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1. Introduction

Hollow and Solid spherical shells are a common type of structure in engineering mechanics. This problem is studied by several researchers in the past. Among them, Srinath [1] obtained the analytical expressions of stresses and displacement in a solid sphere subjected to external pressure. Xiao-Ming and Zong-Da [2] using the method of weighted residuals, obtained the general solutions in forms of Legendre series for thick spherical shell and solid sphere.

Functionally graded materials (FGMs) are a class of new advanced composite materials with continuously varying material properties in one or multi spatial directions and consist of two or more constituents by changing their volume fraction for the goal of optimizing their performance. Closed-form solutions are obtained by Tutuncu and Ozturk [3] for cylindrical and spherical vessels with variable elastic properties obeying a simple power law through the wall thickness which resulted in simple Euler-Cauchy equations whose solutions were readily available. Elastic analysis of internally pressurized thick-walled spherical pressure vessels of functionally graded materials was studied by You et al. [4].

Based on the assumption that Poisson's ratio is constant and modulus of elasticity is an exponential function of radius, Chen and Lin [5] have analyzed stresses and displacements in FG cylindrical and spherical pressure vessels. Singh et al. [6], making use of the particular forms of heterogeneity, solved the equation of equilibrium for torsional vibrations of a solid sphere made of functionally graded materials. A hollow sphere made of FGMs subjected to radial pressure was analyzed by Li et al. [7]. Using plane elasticity theory and Complementary Functions method, Tutuncu and Temel [8] are obtained axisymmetric displacements and stresses in functionally-graded hollow cylinders, disks and spheres subjected to uniform internal pressure. Zamani Nejad et al. [9] developed 3D set of field equations of FGM thick shells of revolution in curvilinear coordinate system by tensor calculus. An analytical solution is obtained by Wei [10] for inhomogeneous strain and stress distributions within solid spheres of Si_{1-x} Ge_x alloy under diametrical compression.

Deformations and stresses inside multilayered thick-walled spheres are investigated by Borisov [11]. In the paper, each sphere is characterized by its elastic modules. Assuming the volume fractions of two phases of a functionally graded (FG) material (FGM) vary only with the radius, Nie et al. [12] obtained a technique to tailor materials for FG linear elastic hollow cylinders and spheres to attain through-the-thickness either a constant circumferential (or hoop) stress or a constant in-plane shear stress. Ghannad and Zamani Nejad [13] presented a complete analytical solution for FGM thick-walled spherical shells subjected to internal and/or external pressures. In another work, Zamani Nejad et al. [14] obtained an exact analytical solution and a numerical solution for stresses and displacements of pressurized thick spheres made of functionally graded material with exponentially-varying properties. On the basis of plane elasticity theory (PET), the displacement and stress components in a thick-walled spherical pressure vessels made of heterogeneous materials subjected to internal and external pressure is developed [15].

In this study, an elastic solution and a numerical solution for pressurized solid sphere made of functionally graded material is presented.

2. Analysis

An axisymmetric solid sphere with radius b is shown in Fig. 1 with the properties changing continuously along radial direction. The sphere is subjected axisymmetric constant pressure P_o on its outer surface.

The problem can be studied in the spherical coordinates (r,θ,ϕ). In this paper, it is assumed that the Poison's ratio υ, takes a constant value and the modulus of elasticity E, is assumed to vary radially according to exponential form as follows [16],

$$\left.\begin{aligned} &E(r)=E_0 e^{-nR^{\eta}};\\ &n=ln\left(\frac{E_0}{E_{out}}\right);\\ &R=\frac{r}{b}, \end{aligned}\right\} \tag{1}$$

where E_0 and E_{out} are modulus of elasticity in center and outer surface, respectively. n and η are material parameters. The displacement in the r-direction is denoted by u. Three strain components can be expressed as:

$$\varepsilon_r = \frac{du}{dr}; \tag{2}$$

$$\varepsilon_\theta = \varepsilon_\phi = \frac{u}{r}, \tag{3}$$

where ε_r and $\varepsilon_\theta = \varepsilon_\phi$ are radial and circumferential strains.

The Hooke's law are given by:

$$\begin{bmatrix} \varepsilon_r \\ \varepsilon_\theta = \varepsilon_\phi \end{bmatrix} = \frac{1}{E}\begin{bmatrix} 1 & -2\upsilon \\ -\upsilon & 1-\upsilon \end{bmatrix}\begin{bmatrix} \sigma_r \\ \sigma_\theta = \sigma_\phi \end{bmatrix}, \tag{4}$$

where σ_r and $\sigma_\theta = \sigma_\phi$ are radial and circumferential stresses.

Substituting Eqs. (2) and (3) into Eq. (4) yields:

$$\begin{pmatrix} \sigma_r \\ \sigma_\theta \end{pmatrix} = \frac{E(R)}{b}\begin{pmatrix} A & 2B \\ B & A+B \end{pmatrix}\begin{pmatrix} \dfrac{du}{dR} \\ \dfrac{u}{R} \end{pmatrix}, \tag{5}$$

where

$$\left.\begin{aligned} A &= \frac{1-\upsilon}{(1+\upsilon)(1-2\upsilon)}; \\ B &= \frac{\upsilon}{(1+\upsilon)(1-2\upsilon)}; \\ \upsilon^* &= \frac{B}{A} = \frac{\upsilon}{1-\upsilon}. \end{aligned}\right\} \tag{6}$$

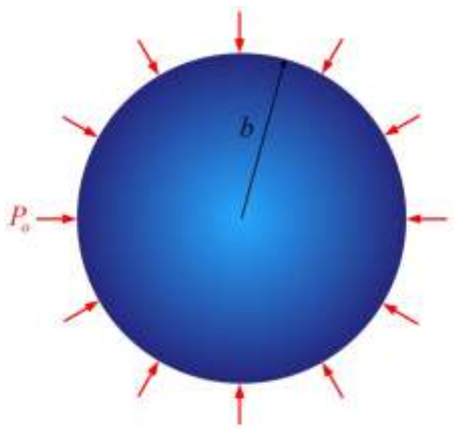

Fig. 1 Geometry of the FGM solid sphere subjected to constant external pressure

The equilibrium equation of the FGM solid sphere, in the absence of body forces, is expressed as:

$$\frac{d}{dR}\left(R^2\sigma_r\right) - 2R\sigma_\theta = 0. \tag{7}$$

Substituting Eq. (5), into Eq. (7), the equilibrium equation is expressed as:

$$R^2\frac{d^2u}{dR^2} + R\left(2 + \frac{RE'}{E}\right)\frac{du}{dR} - 2\left(1 - \upsilon^*\frac{RE'}{E}\right)u = 0, \tag{8}$$

here, prime denotes differentiation with respect to *R*.

The general solution of Eq. (8) is as follows:

$$u(R) = C_1G(R) + C_2H(R), \tag{9}$$

where C_1 and C_2 are arbitrary integration constants, and $G(R)$ and $H(R)$ are homogeneous solutions.

Substituting Eq. (9) into Eq. (5), yields:

$$\begin{pmatrix} \sigma_r \\ \sigma_\theta \end{pmatrix} = \frac{E(R)}{b}\begin{pmatrix} A & 2B \\ B & A+B \end{pmatrix}\begin{pmatrix} C_1G' + C_2H' \\ C_1\dfrac{G}{R} + C_2\dfrac{H}{R} \end{pmatrix}. \tag{10}$$

The forms of $G(R)$ and $H(R)$ will be determined next.

Substituting Eq. (1) into Eq. (8), the governing differential equation is as follows:

$$R^2\frac{d^2u}{dR^2} + R\left(2 - \eta nR^\eta\right)\frac{du}{dR} - 2\left(1 + \upsilon^*\eta nR^\eta\right)u = 0. \tag{11}$$

Eq. (11) is a homogeneous hypergeometric differential equation. Using a new variable $x = nR^\eta$ and applying the transformation $u(R) = Ry(x)$, the result Eq. (11) is:

$$x\frac{d^2y}{dx^2} + \left(1 + \frac{3}{\eta} - x\right)\frac{dy}{dx} - \frac{1+2\upsilon^*}{\eta}y = 0. \tag{12}$$

The solution of Eq. (12) is given as:

$$\begin{aligned} y(x) &= C_1F_C(\alpha,\beta;x) + \\ &+ \bar{C}_2x^{-3/\eta}F_C(\alpha-\beta+1, 2-\beta; x). \end{aligned} \tag{13}$$

In Eq. (13), $F_C(\alpha,\beta;x)$ is the confluent hypergeometric function defined by the series [17]:

$$F_C(\alpha,\beta;x) = 1 + \sum_{k=1}^{\infty}\frac{(\alpha)_k}{(\beta)_k}\frac{x^k}{k!}, \tag{14}$$

where

$$(\alpha)_k = \alpha(\alpha+1)(\alpha+2)\cdots(\alpha+k-1). \tag{15}$$

Thus

$$\begin{aligned} F_C(\alpha,\beta;x) &= 1 + \frac{\alpha}{\beta}\frac{x}{1!} + \frac{\alpha(\alpha+1)}{\beta(\beta+1)}\frac{x^2}{2!} + \\ &+ \frac{\alpha(\alpha+1)(\alpha+2)}{\beta(\beta+1)(\beta+2)}\frac{x^3}{3!} + \cdots \end{aligned} \tag{16}$$

The arguments α, β of F_C in Eq. (16) are determined as:

$$\left.\begin{aligned} \alpha &= \frac{1+2\upsilon^*}{\eta} \\ \beta &= 1 + \frac{3}{\eta} \end{aligned}\right\}. \tag{17}$$

From $u(R)=Ry(nR^{\eta})$, the homogeneous solutions $G(R)$ and $H(R)$ are found in the form:

$$G(R)=RF_C(\alpha,\beta;nR^{\eta}); \tag{18}$$

$$H(R)=\frac{1}{R^2}F_C(\alpha-\beta+1,2-\beta;nR^{\eta}). \tag{19}$$

The Eqs. (9) and (10) may be rewritten with non-dimensional paparmeters as:

$$U(R)=C_3G(R)+C_4H(R); \tag{20}$$

$$\begin{pmatrix}\bar{\sigma}_r\\ \bar{\sigma}_\theta\end{pmatrix}=e^{-nR^{\eta}}\begin{pmatrix}A & 2B\\ B & A+B\end{pmatrix}\begin{pmatrix}C_3G'+C_4H'\\ C_3\dfrac{G}{R}+C_4\dfrac{H}{R}\end{pmatrix}, \tag{21}$$

where

$$\left.\begin{aligned}&U=\frac{uE_0}{bP_o};\\ &\bar{\sigma}=\frac{\sigma}{P_o};\\ &\frac{C_3}{C_1}=\frac{C_4}{C_1}=\frac{E_0}{bP_o}.\end{aligned}\right\} \tag{22}$$

Integration constants C_1 and C_2 are determined by using the following boundary conditions:

$$\left.\begin{aligned}&U(R=0)=0;\\ &\bar{\sigma}_r(R=1)=-1.\end{aligned}\right\} \tag{23}$$

Thus

$$\left.\begin{aligned}&C_3=-\frac{e^n}{AG'(1)+2BG(1)};\\ &C_4=0.\end{aligned}\right\} \tag{24}$$

Hence, non-dimensional radial displacement, radial stress and circumferential stress are found as follows:

$$U=-\frac{e^nG(R)}{AG'(1)+2BG(1)}; \tag{25}$$

$$\bar{\sigma}_r=-e^{n(1-R^{\eta})}\left[\frac{AG'(R)+2B\dfrac{G(R)}{R}}{AG'(1)+2BG(1)}\right]; \tag{26}$$

$$\bar{\sigma}_\theta=-e^{n(1-R^{\eta})}\left[\frac{BG'(R)+(A+B)\dfrac{G(R)}{R}}{AG'(1)+2BG(1)}\right]. \tag{27}$$

4. Numerical analysis

The finite element method is a powerful numerical method in solid mechanics. In this study in order to numerical analysis of problem, a geometry specimen was modeled using a commercial FE code, ANSYS 12, for a comparative study. The numerical solution is done by using of PLANE82 element, and the number of elements and nodes are considered 1371 and 2846, respectively. An axisymmetric element has been applied for modeling and meshing. For modeling of FGM solid sphere, the variation in material properties was implemented by having 20 layers, with each layer having a constant value of material properties.

5. Results and discussion

Consider a solid sphere with an arbitrary radius of b, subjected to an arbitrary constant uniform pressure P_o. It is assumed that the Poisson's ratio υ, has a constant value of 0.3.

For the presentation of the results, use the following dimensionless and normalized variables.

In Fig. 2, for different values of n and η, dimensionless modulus of elasticity along through the radial direction is plotted. It is apparent from the curve that at the same position $(0<R<1)$, for $n=-0.5$, dimensionless modulus of elasticity increases as η decreases, while for $n=+0.5$, the reverse holds true.

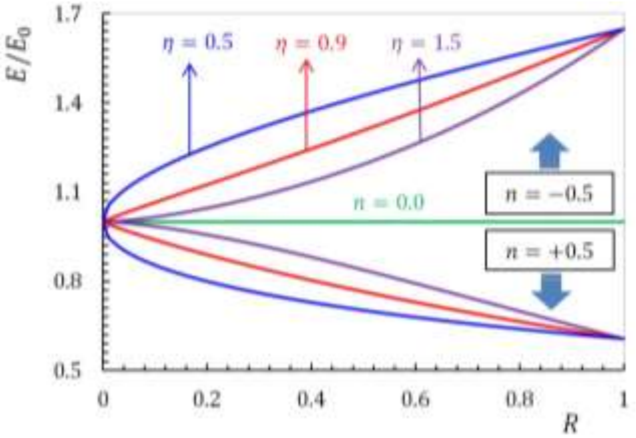

Fig. 2 Radial distribution of Modulus of elasticity

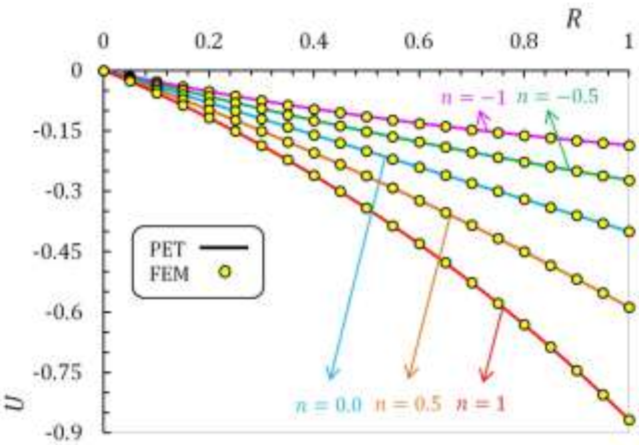

Fig. 3 Radial distribution of radial displacement ($\eta=0.9$)

Distribution of the radial displacement and the radial stress along the radial direction for different values of n and constant value of $\eta=0.9$ are shown in Figs. 3 and 4. According to these figures, at the same position

$(0 < R < 1)$, for higher values of n, radial displacement and radial stress increase.

The circumferential stress along the radial direction for different values of n and constant value of $\eta = 0.9$ is plotted in Fig. 5. It must be noted from this figure that at the same position, almost for $R < 0.65$, there is an increase in the value of the circumferential stress as n increases, whereas for $R > 0.65$ this situation was reversed. Besides, along the radial direction for the positive magnitudes of n the circumferential stress decreases, while for negative magnitude of n, the circumferential stress increases.

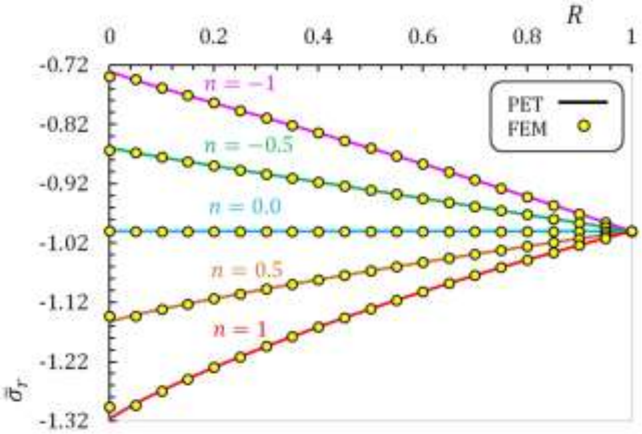

Fig. 4 Radial distribution of radial stress ($\eta = 0.9$)

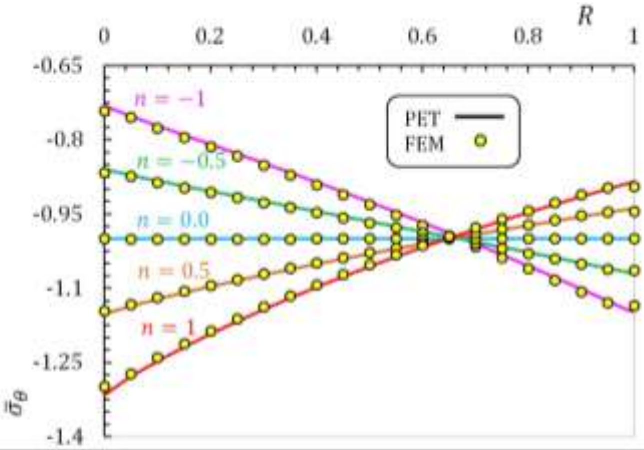

Fig. 5 Radial distribution of circumferential stress ($\eta = 0.9$)

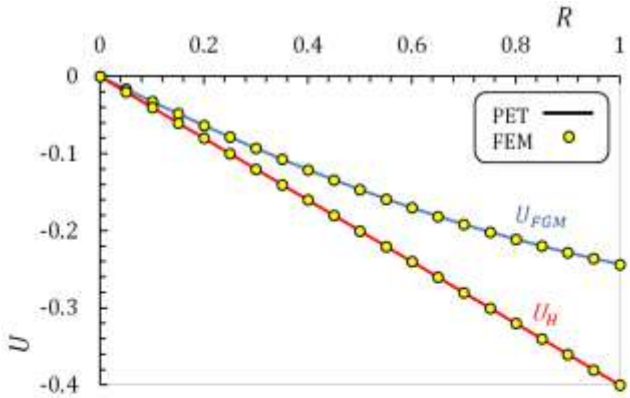

Fig. 6 Comparison of radial displacement in a FGM solid sphere ($n = -0.5$, $\eta = 1.5$) to those in homogeneous solid sphere ($n = 0$)

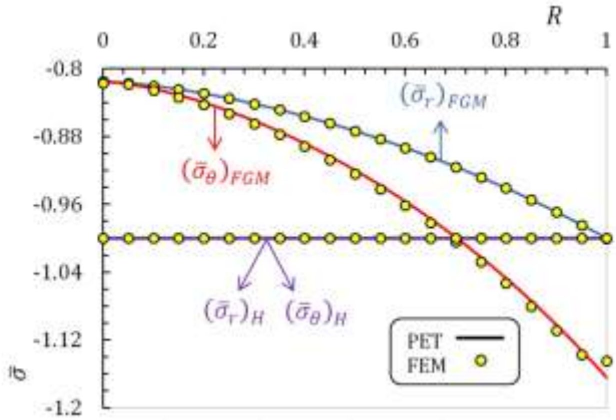

Fig. 7 Comparison of stresses in a FGM solid sphere ($n = -0.5$, $\eta = 1.5$) to those in homogeneous solid sphere ($n = 0$)

Using values $n = -0.5$ and $\eta = 1.5$ for the material parameters, the stresses and displacement in an FGM solid sphere is calculated and compared to those in a homogeneous solid sphere ($n = 0$) in Figs. 6 and 7. The effect of material parameter n on the deformation behavior of the solid sphere is also evaluated.

6. Conclusions

In this work, elastic and numerical solutions for stresses, and displacement in pressurized FGM solid sphere are obtained. The material properties except Poisson's ratio are assumed to be exponential-varying in the radial direction.

To show the effect of inhomogeneity on the stress distributions, different values were considered for material parameter n. Numerical results showed that the inhomogeneity parameter n has great effect on the distributions of elastic fields. For example, the maximum of radial and circumferential stresses for negative values of material parameter n, occur on the external surface, whereas for positive values of n, this situation was reversed. Thus by selecting a proper value of n, it is possible for engineers to design a solid sphere that can meet some special requirements.

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DVIMATĖS TAMPRUMO TEORIJOS TAIKYMAI SFEROS PAGAMINTOS IŠ EKSPONENTIŠKAI FUNKCIONALIAI SLUOKSNIUOTOS MEDŽIAGOS TAMPRUMO ANALIZEI

R e z i u m ė

Šio tyrimo tikslas pateikti analitinį ir skaitmeninį įtemptos sferos įtempių ir radialinių poslinkių vertinimą panaudojant tamprumo plokštumoje teoriją. Sfera sudaryta iš funkcionaliai sluoksniuotos medžiagos. Puasono koeficientas laikomas pastoviu, o tamprumo modulis paklūsta eksponentinei variacijai radialine kryptimi. Šis tyrimas siejamas su medžiagos nehomogeniškumu tampriosioms deformacijoms ir įtempiams. Poslinkių ir įtempių pasiskirstymas palyginamas su sprendimu baigtinių elementų metodu. Atitinkamas skaitmeninis sprendinys rodo, kad pasiūlytas sprendimas yra tikslus ir konverguoja.

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THE AAPLICATION OF 2-DIMENSIONAL ELASTICITY FOR THE ELASTIC ANALYSIS OF SOLID SPHERE MADE OF EXPONENTIAL FUNCTIONALLY GRADED MATERIALS

S u m m a r y

The objective of this study is to present an analytical and a numerical solution for stresses and radial displacement in pressurized solid sphere using plane elasticity theory (PET). The solid sphere composed of functionally graded material (FGM). Poisson’s ratio is assumed constant and modulus of elasticity to obey the exponential variation in the radial direction. The emphasis of this research is laid on effect of material inhomogeneity on the elastic deformation and stresses. The displacement and stresses distributions are compared with the solutions of the finite element method (FEM) and comparison with the corresponding numerical solution indicates that the proposed solution has excellent convergence and accuracy.

Keywords: elastic, solid sphere, exponential, functionally graded material (FGM).

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