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Analysis of uniaxial tension and circumferential inflation on the mechanical property of arterial wall

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1. Introduction

Cardiovascular diseases are one of the major death factors in the modern high civilized world. Therefore much effort is put to the research aimed to explain the mechanisms that govern the cardiovascular system in both healthy and pathological cases [1]. Nonlinear elasticity is now extensively used to study the mechanical response of arterial walls under certain conditions [2-3]. Understanding of arterial wall's fundamental elastic properties and particularly their nonlinear stress–strain characteristics becomes more and more important. From uniaxial tests of arterial wall, we can see that arterial wall has the phenomenon of non-linear stress strain relation and having higher extensibility in the low stress and progressively lower with increasing stretch, which is well known also in the framework of rubber-like materials [4]. Therefore, based finite deformation theory, research on arterial wall by adopting the method of rubber like materials is rational [5, 6]. Many constitutive models have been constructed to describe the physical characteristics of arterial wall based on continuum mechanics. The prototype strain energy for isotropic materials was proposed by Knowles [7] in 1977, which is the popular biomechanical models. The Knowles' power-law strain-energy is given by:

$$W=\frac{\mu}{2b}\left[\left(1+\frac{b}{n}\left(I_1-3\right)\right)^n-1\right], \quad (1)$$

where μ is the shear modulus, b and n are positive material parameters and I_1 the first principal invariants of the Cauchy-Green deformation tensor. When $n>1$ and $n\to\infty$, we can get the Fung's strain energy function [8] as shown in Eq. (2), which was a very popular model in biomechanics:

$$W=\frac{\mu}{2b}\left\{exp\left[b\left(I_1-3\right)\right]-1\right\}. \quad (2)$$

Gent simplified the Fung's strain energy function for incompressible materials. One of the simplest strain-energies functions [9] of Gent for incompressible materials is given by:

$$W=-\frac{\mu}{2}J_m\,ln\left(1-\frac{I_1-3}{J_m}\right), \quad (3)$$

where μ is the shear modulus and J_m is the constant limiting value for I_1-3.

From Eq. (3), Gent's strain energy function can be modified as:

$$W=-\frac{\mu J_m}{2}ln\left(1-\frac{I_1^n-3^n}{J_m}\right), \quad (4)$$

where n is material parameter which can be determined from experiments. In the Modified strain energy function Eq. (4), exponent n has been introduced. When $n = 1$, the constitutive model Eq. (4) can be simplified to the Gent Model. Simultaneously, the exponent n can reflect the mechanical property of arterial wall for different age group.

Based on the elastic finite deformation theory, the Cauchy stress tensor can be expressed as follows:

$$\boldsymbol{\sigma}=-p\boldsymbol{I}+\frac{n\mu J_m}{J_m-\left(I_1^n-3^n\right)}I_1^{n-1}\boldsymbol{B}, \quad (5)$$

where $\boldsymbol{B}=\boldsymbol{F}\boldsymbol{F}^T$ is left Cauchy-Green deformation tensor, I_1 is the first invariants of $\boldsymbol{B}$ and p is the undeterm-ined scalar function that justifies the incompressible internal constraint conditions.

The present work was carried out in order to analyze the uniaxial tension and circumferential inflation on the mechanical property of arterial wall based on the modified strain energy function from Gent. By utilizing the nonlinear finite element software MSC.Marc, numerical simulation on mechanical property of arterial wall was carried out, which illustrate that modified constitutive model describes the finite deformation property of arterial wall reasonably and the applied range has been broadened.

2. Theoretical analysis

For uniaxial tension of cylinder arterial wall, the deformation meets the following expressions:

$$r=f\left(R\right),\quad \theta=\Theta,\quad z=\lambda_z Z, \quad (6)$$

where λ_z is axial principal stretch; r,θ,z are cylindrical coordinate system at current configuration; R,Θ,Z are cylindrical coordinate system at initial configuration.

The deformation gradient and left Cauchy-Green deformation tensor are as follows:

$$\begin{cases} \boldsymbol{F} = \lambda_r \boldsymbol{e}_r \otimes \boldsymbol{E}_R + \lambda_\theta \boldsymbol{e}_\theta \otimes + \lambda_z \boldsymbol{e}_z \otimes \boldsymbol{E}_Z, \\ \boldsymbol{B} = \boldsymbol{F}\boldsymbol{F}^T = \lambda_r^2 \boldsymbol{e}_r \otimes \boldsymbol{E}_R + \lambda_\theta^2 \boldsymbol{e}_\theta \otimes + \lambda_z^2 \boldsymbol{e}_z \otimes \boldsymbol{E}_Z, \end{cases} \tag{7}$$

where λ_r and λ_θ are radial principal stretch and circumferential principal stretch accordingly. For incompressible condition $\lambda_r \lambda_\theta \lambda_z = 1$.

The stress component can be achieved from the cons-titutive Eq. (5) as:

$$\begin{cases} \sigma_{rr} = \sigma_{\theta\theta} = -p + 2\lambda_r^2 \dfrac{\partial W}{\partial I_1}, \\ \sigma_{zz} = -p + 2\lambda_z^2 \dfrac{\partial W}{\partial I_1}. \end{cases} \tag{8}$$

For uniaxial tension, $\sigma_{rr} = \sigma_{\theta\theta} = 0$, we can get:

$$\sigma_{zz} = n\mu J_m \left(\lambda_z^2 - \lambda_z^{-1}\right) \frac{\left(\lambda_z^2 + 2\lambda_z^{-1}\right)^{n-1}}{J_m - \left(\left(\lambda_z^2 + 2\lambda_z^{-1}\right)^n - 3^n\right)}. \tag{9}$$

From Eq. (9) axial force can be expressed as follow:

$$\begin{aligned} F &= \sigma_{zz}\lambda_r^2 = \sigma_{zz}\lambda_z^{-1} = \\ &= n\mu J_m (\lambda_z - \lambda_z^{-2}) \frac{(\lambda_z^2 + 2\lambda_z^{-1})^{n-1}}{J_m - ((\lambda_z^2 + 2\lambda_z^{-1})^n - 3^n)}. \end{aligned} \tag{10}$$

In order to discuss the effect of constitutive parameters J_m and n on the mechanical properties of material, non-dimensional stress is introduced. From the Eq. (9), we can get:

$$F^* = nJ_m(\lambda_z - \lambda_z^{-2}) \frac{(\lambda_z^2 + 2\lambda_z^{-1})^{n-1}}{J_m - ((\lambda_z^2 + 2\lambda_z^{-1})^n - 3^n)}. \tag{11}$$

Figs. 1 and 2 have shown the computed result of Eq. (11), When the parameter n is given ($n = 1$), as shown in Fig. 1. For Fig. 2, as the constitutive parameter J_m increases, the value of stress decreases. For same axial stress, when the constitutive parameter J_m increases, the principal extension ratio becomes larger. On the contrary, if the constitutive parameter J_m decreases, the principal extension ratio becomes smaller. Arterial wall of young people has excellent elasticity, which can adapt dramatic changes of blood pressure of human body. This reveals the new strain energy function from Gent can be used to analyze the deformation of arterial wall under external load. As is shown in Fig. 2. For the given $J_m = 2.289$, if the constitutive parameter n increases, the stress becomes greater and it has the reinforcement feature apparently. Therefore, n is considered as the material's reinforcement parameter.

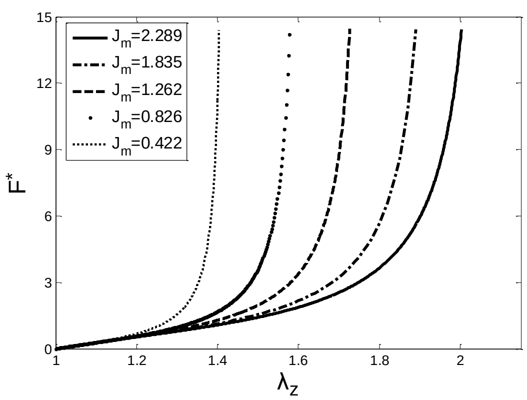

Fig. 1 The relation between $F^* - \lambda_z$ with effect of J_m ($n = 1$)

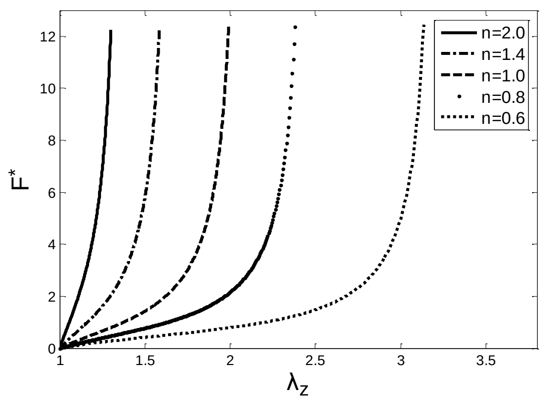

Fig. 2 The relation between $F^* - \lambda_z$ with effect of n ($J_m = 2.289$)

Considering the inflation of cylinder arterial wall, in cylindrical coordinate system the initial geometry of the tube is given by:

$$A \le R \le B, \quad 0 \le \Theta \le 2\pi, \quad 0 \le Z \le L. \tag{12}$$

The deformation of the arterial wall can be expressed as:

$$r = f(R), \ \theta = \Theta, \quad z = \lambda_z Z. \tag{13}$$

The deformation gradient and left Cauchy-Green deformation tensor are shown as follows:

$$\begin{cases} \boldsymbol{F} = \dfrac{dr}{dR} \boldsymbol{e_r} \otimes \boldsymbol{E_R} + \dfrac{1}{R} r \boldsymbol{e}_\theta \otimes \boldsymbol{E}_\theta + \lambda_z \boldsymbol{e_z} \otimes \boldsymbol{E_Z}, \\ \boldsymbol{B} = \left(\dfrac{dr}{dR}\right)^2 \boldsymbol{e_r} \otimes \boldsymbol{E_R} + \left(\dfrac{r}{R}\right)^2 \boldsymbol{e}_\theta \otimes \boldsymbol{E}_\theta + \lambda_z^2 \boldsymbol{e_z} \otimes \boldsymbol{E_Z}. \end{cases} \tag{14}$$

From Eq. (14) we can get:

$$\lambda_1 = dr / dR, \quad \lambda_2 = r / R, \quad \lambda_3 = \lambda_z. \tag{15}$$

Let $\lambda_2 = r / R = \lambda$ and the incompressible condition $\lambda_1 \lambda_2 \lambda_3 = 1$ we can get:

$$\lambda_1 = (\lambda\lambda_z)^{-1}, \quad \lambda_2 = \lambda, \quad \lambda_3 = \lambda_z, \tag{16}$$

$$I_1 = trB = (\lambda\lambda_z)^{-2} + \lambda^2 + \lambda_z^2. \tag{17}$$

Substituting the Eqs. (14), (16) and (17) to Eq. (5), the following expressions can be achieved:

$$\begin{cases} \sigma_{rr} = -p + 2W_1(\lambda\lambda_z)^{-2}, \\ \sigma_{\theta\theta} = -p + 2W_1\lambda^2, \\ \sigma_{zz} = -p + 2W_1\lambda_z^2, \end{cases} \tag{18}$$

where $W_1 = \partial W / \partial I_1$.

For free inflation, $\sigma_{zz} = 0$. Eq. (18) becomes as:

$$\begin{cases} \sigma_{rr} = 2W_1\left[(\lambda\lambda_z)^{-2} - \lambda_z^2\right], \\ \sigma_{\theta\theta} = 2W_1(\lambda^2 - \lambda_z^2). \end{cases} \tag{19}$$

In the absence of body forces the equilibrium equation is expressed as $div\boldsymbol{T}$ = 0. From the equilibrium equation, only one component is not satisfied identically, namely, the radial component, which is:

$$\frac{d\sigma_{rr}}{dr} + \frac{1}{r}(\sigma_{rr} - \sigma_{\theta\theta}) = 0 \tag{20}$$

We consider the boundary conditions corresponding to a positive pressure p on the inside of the arterial wall and no load on the outside:

$$\sigma_{rr}(a) = -P \ \ (P>0), \ \sigma_{rr}(b) = 0, \tag{21}$$

where P is internal pressure.

Substituting the Eq. (19) to Eqs. (20) and by utilizing the Eq. (21), we can get:

$$P = \frac{n\mu J_m}{J_m - (I_1^n - 3^n)} I_1^{n-1}\left[\lambda^2 - (\lambda\lambda_z)^{-2}\right]\frac{\varepsilon}{\lambda^2\lambda_z}, \tag{22}$$

where $\varepsilon = (B-A)/A$.

In order to discuss the effect of constitutive parameters μ and ε on the mechanical properties of arterial wall, non-dimensional pressure is introduced. From the Eq. (22), we can get:

$$P^* = \frac{nJ_m}{J_m - (I_1^n - 3^n)} I_1^{n-1}\left[\lambda^2 - (\lambda\lambda_z)^{-2}\right]\frac{1}{\lambda^2\lambda_z} \tag{23}$$

where $P^* = P/(\mu\varepsilon)$.

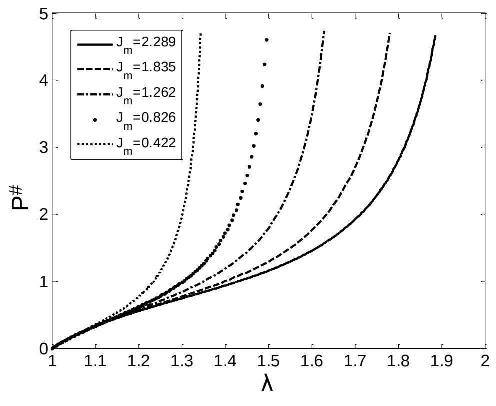

Fig. 3 The relation between $P^* - \lambda$ with effect of J_m ($n=1$, $\lambda_z = 1$)

In order to discuss the influences of constitutive parameters J_m, λ_z and n on the mechanical properties of arterial wall, the Eq. (23) is calculated. The result is shown in Figs. 3 and 4.

When the parameter n and λ_z is given ($n=1$, $\lambda_z=1$), the relation between $P^* - \lambda$ with effect of J_m as illustrated in Fig. 3. If the constant limiting value J_m increases, the rage of circumferential principal stretch of arterial wall is enlarged obviously, which indicates that arterial wall has strong inflation ability and has good toughness. This can reflect the conditon of young people's arterial wall. In the contrary, if the constant limiting value J_m decreases, the rage of circumferential principal stretch of arterial wall becomes nerrow, which reflects the hardening of arterial wall. This can reflect the conditon of old people's arterial wall.

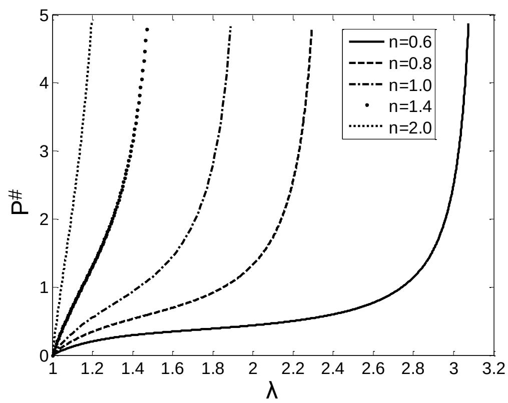

Fig. 4 The relation between $P^* - \lambda$ with effect of n ($J_m = 2.289$, $\lambda_z = 1$)

As the parameter J_m and λ_z is given ($J_m = 2.289$, $\lambda_z = 1$), the relation between $P^* - \lambda$ with effect of n as illustrated in Fig. 4. If the constitutive parameter n increases, the circumferential principal stretch of arterial wall becomes smaller under same internal pressure, which means the inflation ratio of arterial wall decreases correspondingly. This can reflect the conditon of young people's arterial wall. When the value of n decreases, the rage of circumferential principal stretch of arterial wall is

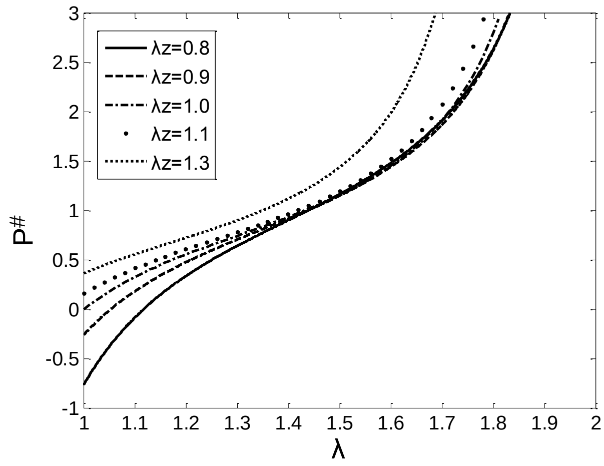

Fig. 5 The relation between $P^* - \lambda$ with effect of λ_z ($J_m = 2.289$, $n=1$)

enlarged, which means that arterial wall has strong inflation ability. This can reflect the conditon of young people's arterial wall. Comparing Fig. 2 with Fig. 1, it can be seen that constitutive parameter n has more effect on the circumferential principal stretch of arterial wall. Therefore, n is considered as the material's reinforcement parameter.

Fig. 5 shows the relation between $P^* - \lambda$ with effect of λ_z . When the parameter J_m and n is given ($J_m = 2.289$, $\lambda_z = 1$). When the value of axial principal stretch of arterial wall λ_z increases, the the circumferential principal stretch of arterial wall becomes smaller under same internal pressure. In the same, if the the value of axial principal stretch of arterial wall λ_z decreases, the the circumferential principal stretch of arterial wall becomes larger. It is consisitent with he incompressible condition of material from arterial wall.

3. Numerical simulation

Finite element analysis on nonlinear elastic deformation of arterial wall have been proposed from [10-12]. The FEA software MSC.Marc is employed in the numerical simulation of this research. The strain energy function of arterial wall was assumed to be the strain energy function of Eq. (4). In order to implement the modified strain energy function from Gent into the finite element procedure, non-linear finite element analysis of arterial wall was performed by a user subroutine when defining the material properties, which allows the users to define the derivatives of the strain energy functions with respect to either the strain invariants or the principal stretches. In the subroutine, w1, w2 and w3 are the first derivatives of the energy function with respect to strain invariants and w11, w22, w33, ww12, ww23, w31 are the second derivatives of the energy function with respect to the strain invariants.

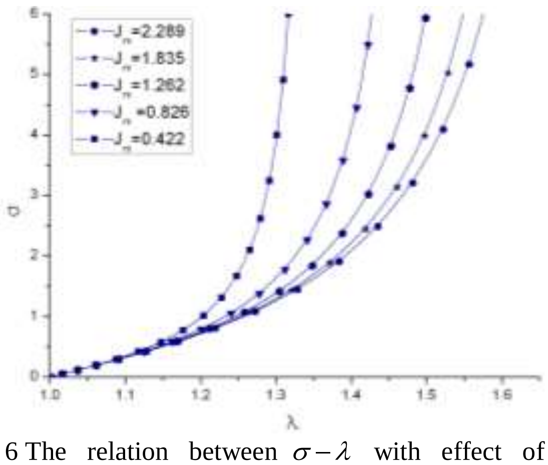

Fig. 6 The relation between $\sigma - \lambda$ with effect of J_m ($n = 1$) from FEM

First, we think about the finite element analysis of uniaxile tension of arterial wall. From [13], for adults inner diameter of arterial wall is 10-30 mm, thickness is 2-3 mm. From that we take 24 mm for outer diameter of arterial and 20 mm for inner diameter. So the thichness of arterial wall is 2 mm. According to the symmetry of structure, a quarter model of arterial wall is established. By fixing constitutive parameter $n = 1$ and changing the value of constitutive parameter J_m , the relation between axial tension stress and axial pincipal stretch is shown as Fig. 6. From which, we can get for same axial stress, when the constitutive parameter J_m increases, the axial principal extension ratio becomes larger. Similarly, when fixing $J_m = 2.289$ and changing the value of the constitutive parameter n , with the constitutive para-meter n increases, the stress becomes greater shown as Fig. 7. This agrees with the theoretical results shown previously.

Arterial wall under pressure is analyzed by utilizing non-linear finite element method. A quarter finite element models is also considered as shown in Fig. 8.

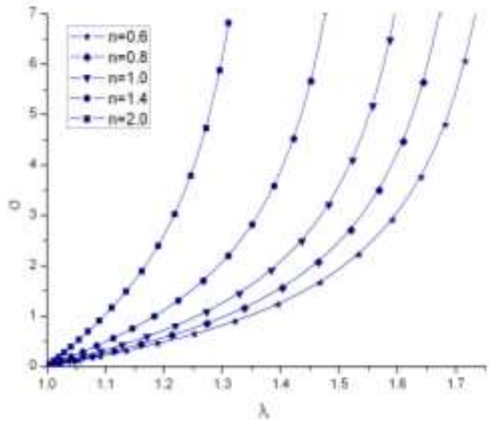

Fig. 7 The relation between $\sigma - \lambda$ with effect of n ($J_m = 2.289$) from FEM

Fig. 8 The finite element model of arterial wall under internal pressure

Figs. 9 and 10 show the stress contour of arterial wall under internal pressure. From that, we can see the radial stress of arterial wall is compress stress. The absolute value of radial stress increases with the increase of pressure along the radius. At inside of arterial wall, radial stress change obviously, but at outside of arterial wall, radial stress entirely approachs to zero. Circumferential stress is tensile stress, which also increases with the increase of pressure along the radius. The maxmum circumferential stress appears at inside wall and minimum circumferential stress appears outside wall, value of which is all greater than radial stress at same internal pressure. The relation between pressure and circumferential pincipal stretch from non-linear finite element analysis is shown as Figs. 11 and 12. From which we can get, when the parameter λ_z and n is given, as the value of J_m increases, the circumferential principal stretch of arterial wall becomes

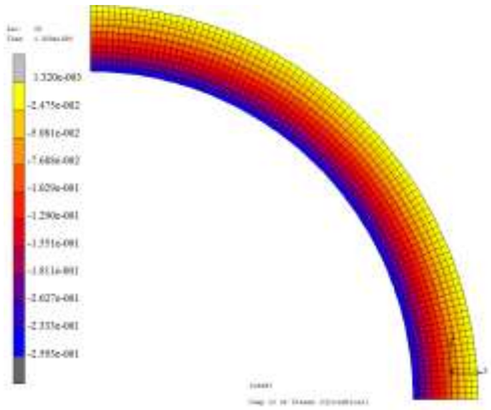

Fig. 9 Radial stress distribution contour

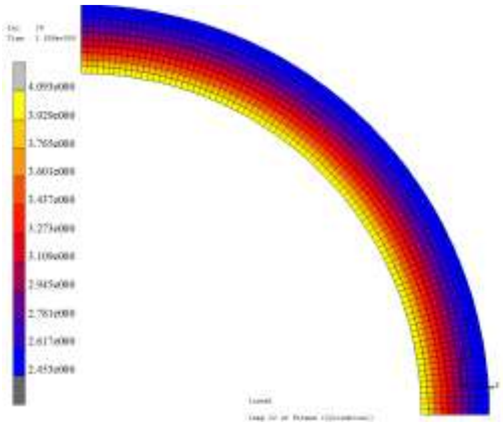

Fig. 10 Circumferential stress distribution contour

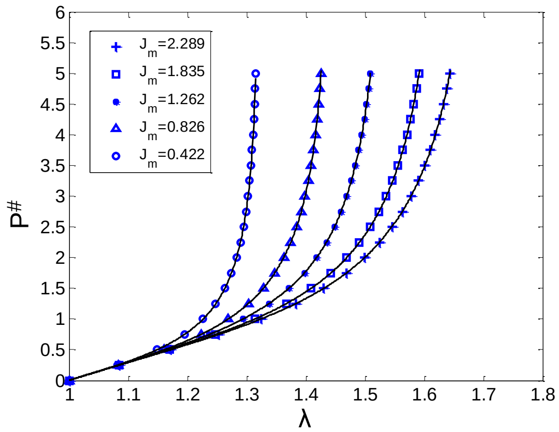

Fig. 11 The relation between $P^{*}-\lambda$ with effect of J_m ($n=1$, $\lambda_z=1$) from FEM

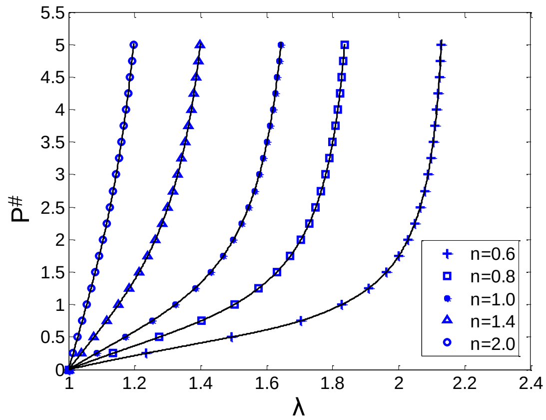

Fig. 12 The relation between $P^{*}-\lambda$ with effect of n ($J_m=2.289$, $\lambda_z=1$) from FEM

smaller under same internal pressure. Similarly as the parameter J_m and λ_z is given. When the value of constitutive parameter λ_z increases, the the circumferential principal stretch of arterial wall becomes smaller under same internal pressure. This also agrees with the theoretical results shown previously.

4. Conclusion

This paper presents the analysis of uniaxial tension and circumferential inflation on the mechanical property of arterial wall. Based on the finite deformation theory. Non-linear elastic analysis of uniaxial tension and inflation of arterial wall has been proposed. By using the finite element software MSC.Marc, mechanical property of arterial wall has been also analyzed. It is found that the new constitutive model fulfills the requirement that the modified strain energy function will transform into Gent model with $n=1$. When $n=1$ and $J_m\to\infty$, the modified strain energy density function can be transformed into neo-hooken model. The constitutive parameters J_m can be considered as ultimate elongation limit parameter of incompressible materials and n can be considered as the material's reinforcement parameter. Both constitutive parameters J_m and n can reflect the conditon of young people and old people's arterial wall. The discussions illustrate that modified constitutive model describes the finite deformation property of arterial wall reasonably and the applied range has been broadened by using the modified constitutive model.

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ANALYSIS OF UNIAXIAL TENSION AND CIRCUMFERENTIAL INFLATION ON THE MECHANICAL PROPERTY OF ARTERIAL WALL

S u m m a r y

Considering the incompressible strain hardening materials, a new strain energy density function is modified from Gent Model, which fulfills the requirement that the modified strain energy function will transform into Gent model with $n=1$. When $n=1$ and $J_m \to \infty$, the modified strain energy density function can be transformed into neo-hooken model. Arterial wall was considered as incompressible hyper-elastic materials. Uniaxial tension and inflation of arterial wall was been researched by utilizing the modified strain energy function. Influence of constitutive parameters on the mechanical property of arterial wall has been researched. In order to verify the theoretical theory, non-linear finite element analysis of arterial wall was proposed. By utilizing the commercial finite element software MSC.Marc, numerical simulation on mechanical property of arterial wall was carried out based on the modified strain energy function, which showed the finite element results agree with the theoretical results properly. The discussions illustrate that new constitutive model describes the finite deformation property of arterial wall reasonably and the applied range has been broadened by using the new constitutive model.

Keywords: Arterial wall, nonlinear elastic analysis, constitutive model, numerical simulation，mechanical property.

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