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Interaction of bridged cracks in a circular disk with mixed boundary conditions

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1. Introduction

Circular disks are widely used in contemporary machines. The disk strength issues are very urgent, and undoubtedly the interest to these issues will grow in connection with the existing tendency of development of engineering and energetics. At the design stage of a disk it is necessary to take into account that there may happen crack initiation in the disk, to perform limit analysis of the disk to establish that the adversely located would-be initial cracks will not grow to critical sizes and will not cause fracture in the course of the estimated lifetime. The size of the initial minimal crack should be considered as a design characteristics of the disk material. Extensive references have been devoted to strength analysis of disks [1, 2]. In a great majority of the existing papers A. Griffits's model of a crack is used. In the present paper we use a model of a bridged crack [3-5].

2. Problem statement

We consider a plane problem of fracture mechanics for a circular disk weakened by bridged cracks. We study a quasistatic deformation process of a disk whose cross section in the plane $x+iy$ occupies a circle of radius R (Fig.1). A model of a bridged crack is used. This model of a crack has got the experimental confirmation [3, 6-8].

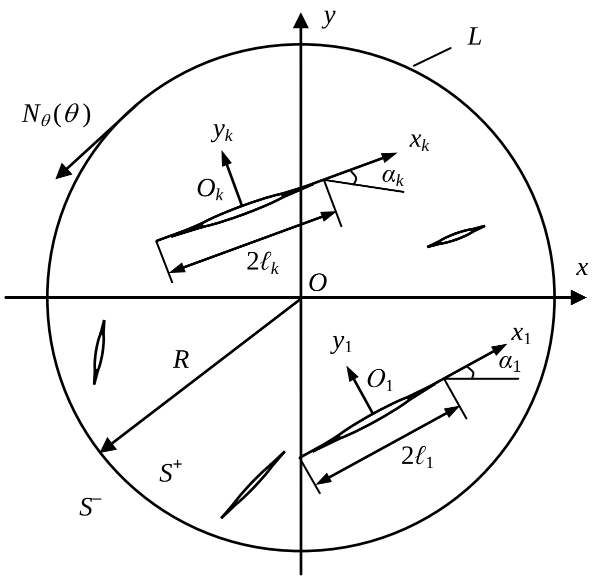

Fig. 1 Computatinal diagram of fracture mechanics problem for circular disk

Let us consider a fracture mechanics problem for a circular disk when the mixed boundary conditions are given on the contour of the disk. Refer the cross section of the disk to polar system of coordinates $r\theta$ having chosen the origin of coordinates at the center of the circle L of radius R.

Let only normal displacements $u_r(\theta)$ and tangential components of the surface force $N_\theta(\theta)$ be given on the contour of the disk. The disk has N rectilinear cracks of length $2l_{0k}$ $(k=1,2,\ldots,N)$. Locate at the center of the crack the origin of local system of coordinates $x_kO_ky_k$ whose axes x_k coincide with the lines of the cracks and form the angles α_k with the axis x (Fig. 1).

We consider a crack model with cohesive forces (bonds) continuously distributed in narrow end zones of the cracks and having the given deformation diagram. The cohesive forces (bonds) will be concentrated in the narrow domains D_k, the sizes of these domains are unknown beforehand and should be defined from the problem solution.

It is accepted that the fracture process for each crack is localized at the end zone that is considered as a part of the crack and may be compared with the crack size. In the studied case, the fracture process zone may be considered as some layer (end zone) adjacent to the crack and containing a material with partially disturbed bonds between its separate structural elements. When the disk is loaded, in the bonds of cracks at the end zones there will arise normal $q_{y_k}(x_k)$ and tangential $q_{x_ky_k}(x_k)$ forces $(k=1,2,\ldots,N)$. The quantity stresses and sizes d_{1k} and d_{2k} of these end zones are unknown beforehand and should be determined.

As the end zones are small compared with the remaining part of the disk, they can be removed mentally having changed by the sections that interact by some low corresponding to the action of the removed material. Thus, to the crack faces at the end zones will apply the normal and tangential stresses numerically equal to $q_{y_k}(x_k)$ and $q_{x_ky_k}(x_k)$ $(k=1,2,\ldots,N)$, respectively.

Denote by $L'=\sum_{k=1}^{N}L'_k$ the set of free faces of the cracks, and by $L''=\sum_{k=1}^{N}L''_k$ the set of prefracture end zones at which the faces interact with bonds. The boundary conditions of the problem on the faces of cracks with end zones have the form:

$$\left.\begin{aligned}&\sigma_{y_k}=0 \quad \tau_{x_k y_k}=0 \quad \text{on} \quad L'_k \quad (k=1,2,\ldots,N),\\ &\sigma_{y_k}=q_{y_k}(x_k) \quad \tau_{x_k y_k}=q_{x_k y_k}(x_k) \quad \text{on} \quad L''_k.\end{aligned}\right\} \quad (1)$$

The main relations of the problem should be complemented with the equations connecting the opening of the end zones faces and forces in the bonds. Without loss of generality these equations are represented in the form [4].

$$\left.\begin{aligned}&\left(v_k^+-v_k^-\right)-i\left(u_k^+-u_k^-\right)=\Pi\left(x_k,\sigma_k\right)\left[q_{y_k}(x_k)-iq_{x_k y_k}(x_k)\right],\\ &\sigma_k=\sqrt{q_{y_k}^2+q_{x_k y_k}^2}\,.\end{aligned}\right\} \quad (2)$$

The functions $\Pi(x_k\sigma_k)$ represents effective compliances of bonds dependent on tensions; σ_k are the stress vector module in the bonds; $\left(v_k^+-v_k^-\right)$ is a normal, $\left(u_k^+-u_k^-\right)$ is a tangential component of the opening of the end zone faces of the k-th crack.

We denote the domain under consideration enclosed between the circle L of radius R and the system of sections $L_k=[-l_k,l_k]$ $(k=1,2,\ldots,N)$ by S^+, the domain supplemented to the complete complex plane by S^-.

The problem is reduced to determination of two complex variable functions $\Phi(z)$ and $\Psi(z)$ analytic in the domain S^+ and satisfying the boundary conditions [9]:

$$Re\left\{\kappa\Phi(t)-\overline{\Phi(t)}+\frac{R^2}{t^2}\left[t\overline{\Phi'(t)}+\overline{\Psi(t)}\right]\right\}=$$

$$=2\mu u'_r(t) \quad \text{on } L, \qquad (3)$$

$$Im\left\{\Phi(t)+\overline{\Phi(t)}-\frac{t^2}{R^2}\left[\bar{t}\Phi'(t)+\Psi(t)\right]\right\}=$$

$$=-N_\theta(t) \quad \text{on } L, \qquad (4)$$

$$\Phi(x_k)+\overline{\Phi(x_k)}+x_k\overline{\Psi'(x_k)}+\overline{\Psi(x_k)}=F_k$$

$$(k=1,2,\ldots,N), \qquad (5)$$

$$F_k=\begin{cases}0 & \text{on} \quad L'_k\\ q_{y_k}-iq_{x_k y_k} & \text{on} \quad L''_k\end{cases}$$

where $\tau=Re^{i\theta}$; x_k are the affixes of the points of faces of the k-th crack with end zones; κ is the Muskhelishvili constant; μ is the shear modulus of the disk material.

On the circle L in the general case we take the functions $u_r(\tau)$ and $N_\theta(\tau)$ in the form of the Fourier series

$$u_r(\tau)=\sum_{\nu=-\infty}^{\infty}V_\nu\left(\frac{\tau}{R}\right)^\nu, \qquad iN_\theta=\sum_{\nu=-\infty}^{\infty}T_\nu\left(\frac{\tau}{R}\right)^\nu,$$

where V_ν, T_ν ($\nu=0,\pm1,\pm2,\ldots$) are, generally speaking, the known complex coefficients.

3. Method of the boundary-value problem solution

Passing in relations (3) and (4) to conjugate values, after some transformations on the contour L we get the following relation

$$(\kappa-1)\left[\Phi(\tau)+\overline{\Phi(\tau)}\right]+\frac{2R^2}{\tau^2}\left[\frac{R^2}{t}\overline{\Phi'(\tau)}+\overline{\Psi(\tau)}\right]=$$
$$=2\left[2\mu u'_r(\tau)+iN_\theta(\tau)\right]. \qquad (6)$$

Introduce on L a new auxiliary function $\omega(t)\in H$ (the Holder condition) in the form

$$2\omega(\tau)=(\kappa-1)\left[\Phi(\tau)-\overline{\Phi(\tau)}\right]-$$
$$-\frac{2\tau^2}{R^2}\left[\frac{R^2}{\tau}\Phi'(\tau)+\Psi(\tau)\right] \quad \text{on } L. \qquad (7)$$

Summing up (6) and (7), we find

$$\Phi(\tau)=\frac{1}{\kappa-1}\left[\omega(\tau)+2\mu u'_r(\tau)+iN_\theta(\tau)\right]. \qquad (8)$$

Now, having substituted (8) in (7), we get

$$\Psi(\tau)=Q(\tau)+R_1(\tau)+R_2(\tau) \quad \text{on } L. \qquad (9)$$

where

$$Q(t)=-\frac{R^2}{2t^2}\left[\omega(t)+\overline{\omega(t)}\right]-\frac{R^2}{(\kappa-1)t}\omega'(t),$$

$$R_1(t)=\sum_{\nu=0}^{\infty}\left[\frac{1}{2}\left(1-\frac{\nu-2}{\kappa-1}\right)T_{\nu+2}-\frac{1}{2}\overline{T}_{-\nu-2}+\frac{\mu(\nu+1)}{R}V_{-\nu-1}+\frac{\mu(\nu+3)}{R}\left(1-\frac{\nu-2}{\kappa-1}\right)V_{\nu+3}\right]\left(\frac{t}{R}\right)^\nu,$$

$$R_2(t)=-\sum_{\nu=2}^{\infty}\left[\frac{\mu(\nu-1)}{R}\overline{V}_{\nu-1}+\frac{1}{2}\overline{T}_{\nu-2}\right]\left(\frac{R}{t}\right)^\nu+\sum_{\nu=3}^{\infty}\left[\frac{1}{2}\left(1+\frac{\nu-2}{\kappa-1}\right)T_{-\nu+2}-\right.$$
$$\left.-\frac{\mu(\nu-3)}{R}\left(1+\frac{\nu-2}{\kappa-1}\right)V_{-\nu+3}\right]\left(\frac{R}{t}\right)^\nu+\left[\frac{\mu}{R}V_1+\frac{1}{2}T_0\right]\frac{R^2}{t^2}+\left[\frac{1}{2}\left(1-\frac{1}{\kappa-1}\right)T_1-\frac{1}{2}\overline{T}_{-1}-\frac{\mu}{(\kappa-1)R}V_1+\frac{2\mu}{R}V_2\right]\frac{R}{t}.$$

Based on the theory on analytic continuation and the property of the Cauchy-type integral, from relations (8) and (9) allowing for expansions of the functions $u_r(t)$ and $iN_\theta(t)$ we have

$$\Phi_*(z)=\begin{cases}\Phi(z)-\dfrac{1}{\kappa-1}\left\langle\dfrac{1}{2\pi i}\displaystyle\int_L\dfrac{\omega(t)dt}{t-z}-\dfrac{1}{\kappa-1}\times\right.\\ \left.\times\displaystyle\sum_{k=0}^{\infty}\left[\dfrac{2\mu(k+1)}{R}V_{k+1}+T_k\right]\left(\dfrac{z}{R}\right)^k\right\rangle \quad z\in S^+\\ -\dfrac{1}{\kappa-1}\left\langle\dfrac{1}{2\pi i}\displaystyle\int_L\dfrac{\omega(t)dt}{t-z}+\dfrac{1}{\kappa-1}\times\right.\\ \left.\times\displaystyle\sum_{k=1}^{\infty}\left[T_{-k}-\dfrac{2\mu(k-1)}{R}V_{-k+1}\right]\left(\dfrac{R}{z}\right)^\nu\right\rangle \quad z\in S^-\end{cases} \tag{10}$$

$$\Psi_*(z)=\begin{cases}\Psi(z)-\dfrac{1}{2\pi i}\displaystyle\int_L\dfrac{Q(t)dt}{t-z}-R_1(z) & z\in S^+\\ -\dfrac{1}{2\pi i}\displaystyle\int_L\dfrac{Q(t)}{t-z}dt+R_2(z) & z\in S^-\end{cases}. \tag{11}$$

In relations (10)-(11) the functions $\Phi_*(z)$ and $\Psi_*(z)$ are analytic in the complete complex plane cut along the sections $L_k=[-l_k,l_k]$ $(k=1,2,\dots,N)$ and vanish at infinity, i.e. $\Phi_*(\infty)=0$, $\Psi_*(\infty)=0$.

We will look for the auxiliary unknown function $\omega(t)\in H$ on L in the form

$$\omega(\tau)=\alpha_0^*+\sum_{\nu=1}^{\infty}\left[\alpha_\nu^*\left(\frac{\tau}{R}\right)^\nu+\alpha_{-\nu}^*\left(\frac{R}{t}\right)^\nu\right], \tag{12}$$

where α_ν^* ($\nu=0,\pm1,\pm2,\dots$) are the unknown complex coefficients.

Substituting relation (12) to the first formulas (10) and (11) and using the Cauchy integral theorem, we get general formulas for the desired functions:

$$\Phi(z)=\Phi_*(z)+\Phi_0(z),$$

$$\Psi(z)=\Psi_*(z)+\Psi_0(z),$$

$$\Phi_0(z)=\sum_{k=0}^{\infty}J_k\left(\frac{z}{R}\right)^k,$$

$$\Psi_0(z)=\sum_{k=0}^{\infty}W_k\left(\frac{z}{R}\right)^k,$$

$$J_\nu=\frac{1}{\kappa-1}\left[\alpha_\nu^*+T_k+\frac{2\mu(k+1)}{R}V_{k+1}\right],$$

$$W_k=-\left(\frac{1}{2}+\frac{k+2}{\kappa-1}\right)\alpha_{k+2}^*-\frac{1}{2}\overline{\alpha}_{-k-2}+\frac{1}{2}\left(1-\frac{k+2}{\kappa-1}\right)T_{k+2}-$$
$$-\frac{1}{2}\overline{T}_{-k-2}+\frac{\mu(k+3)}{R}\left(1-\frac{k+2}{\kappa-1}\right)V_{k+3}+\frac{\mu(k+1)}{R}\overline{V}_{-k-1}.$$

We will look for the functions $\Phi_*(z)$ and $\Psi_*(z)$ in the form [10]

$$\left.\begin{aligned}\Phi_*&=\frac{1}{2\pi}\sum_{k=1}^{N}\int_{-l_k}^{l_k}\frac{g_k(t)dt}{t-z_k},\\ \Psi_*(z)&=\frac{1}{2\pi}\sum_{k=1}^{N}e^{-2i\alpha_k}\int_{-l_k}^{l_k}\left[\frac{\overline{g_k(t)}}{t-z_k}-\frac{\overline{T}_k e^{il_k}}{(t-z_k)^2}g_k(t)\right]dt,\end{aligned}\right\} \tag{14}$$

$$T_1=te^{i\alpha_k}+z_k^0,$$
$$z_k^0=x_k^0+iy_k^0,$$
$$z_k=e^{-i\alpha_k}\left(z-z_k^0\right),$$
$$\frac{i(\kappa+1)}{2\mu}g_k(x_k)=\frac{\partial}{\partial x_k}\left[u_k^+-u_k^-+i\left(v_k^+-v_k^-\right)\right]$$
$$(k=1,2,\dots,N).$$

The boundary conditions on the sections $y_k=0$, $-\ell_k\le x_k\le\ell_k$ $(k=1,2,\dots,N)$ are used to find the unknown functions $g_k(x_k)$.

In what follows, we will refer all linear sizes to the radius *R*.

Satisfying boundary conditions (5) by the functions (13)-(14) on the faces of cracks with end zones, we get a system of singular integral equations with respect to the unknown functions $g_k(x_k)$ $(k=1,2,\dots,N)$.

To the system of singular integral equations for the internal cracks with end zones we should add additional conditions following from the physical sense of the problem

$$\int_{-\ell_k}^{\ell_k}g_k(t)dt=0 \qquad (k=1,2,\dots,N). \tag{15}$$

For converting the system of integral equations to an algebraic system, at first by means of change of variables in the system and in conditions (15) we reduce all the integration integrals to one interval [-1;1]. Using the procedure for converting [10-12] the system of integral equations to an algebraic system, we find that to the system of integral equations under conditions (15) there corresponds the following system of $N\times M$ algebraic equations of $N\times M$ unknowns $g_n^0(t_m)$

$$\left.\begin{aligned}&\frac{1}{M}\sum_{m=1}^{M}\sum_{k=1}^{N}\ell_k\left[g_k^0(t_m)K_{nk}(\ell_k t_m,\ell_n x_r)+\right.\\ &\left.+\overline{g_k^0(t_m)}L_{nk}(\ell_k t_m,\ell_n x_r)\right]=F_n(x_r)+F(x_r),\\ &\sum_{m=1}^{M}g_n^0(t_m)=0 \quad (n=1,2,\dots,N;\ r=1,2,\dots,M\text{-}1).\end{aligned}\right\} \tag{16}$$

$$K_{nk}(t,x)=\frac{e^{i\alpha_k}}{2}\left(\frac{1}{T_k-X_n}+\frac{e^{-2i\alpha_n}}{\overline{T}_k-\overline{X}_n}\right),$$

$$L_{nk}(t,x)=\frac{e^{-i\alpha_k}}{2}\left[\frac{1}{\overline{T}_k-\overline{X}_n}-\frac{T_k-X_n}{\left(\overline{T}_k-\overline{X}_n\right)^2}e^{-2i\alpha_n}\right],$$

$$F(x)=-\left[\Phi_0(x)+\overline{\Phi_0(x)}+x\overline{\Phi_0'(x)}+\overline{\Psi_0(x)}\right],$$

$$t_m=cos\frac{2m-1}{2M}\pi \qquad m=1,2,\dots,M,$$

$$x_r=cos\frac{\pi r}{M} \qquad r=1,2,\dots,M-1.$$

If in (16) we pass to conjugate values, we get one more $N\times M$ algebraic equations. After some transformations we represent relation (2) in the form

$$\left.\begin{aligned}&-\frac{1+\kappa}{2\mu}\int_{-l_k}^{x_k} v_k^0(x_k)dx_k = \Pi(x_k,\sigma_k)q_{y_k}(x_k),\\&-\frac{1+\kappa}{2\mu}\int_{-l_k}^{x_k} u_k^0(x_k)dx_k = \Pi(x_k,\sigma_k)q_{x_k y_k}(x_k),\\&(k=1,2,\dots,N),\end{aligned}\right\} \quad (17)$$

where $v_k^0 = \frac{\partial}{\partial x_k}\left(v_k^+ - v_k^-\right)$; $u_k^0 = \frac{\partial}{\partial x_k}\left(u_k^+ - u_k^-\right)$.

The right hand sides of the system (16) contain unknown values of stresses $q_{y_k}(t_{m,k})$ and $q_{x_k y_k}(t_{m,k})$ at the nodal points $t_{m,k}$ (m=1,2,…, $M_{1,k}$; k=1,2,…,N), for constructing the missing equations we require that the conditions (17) at the nodal points $t_{m,k}$, contained at the prefracture end zones be fulfilled.

As a result, we get one more $2\times N$ system from $M_{1,k}$ equations to equations for determining approximate values of $q_{y_k}(t_{m,k})$ and $q_{x_k y_k}(t_{m,k})$ (k=1,2,…,N; m=1,2,…, $M_{1,k}$):

$$\left.\begin{aligned}&Dv_k^0(t_{1,k}) = \Pi(t_{1,k},\sigma_k(t_{1,k}))q_{y_k}(t_{1,k}),\\&D\left(v_k^0(t_{1,k})+v_k^0(t_{2,k})\right) = \Pi(t_{2,k},\sigma_k(t_{2,k}))q_{y_k}(t_{2,k}),\\&\dots\dots\dots\dots\dots\dots,\\&D\sum_{m=1}^{M_{1,k}} v_k^0(t_{m,k}) = \Pi\left(t_{M_{1,k},k},\sigma_k\left(t_{M_{1,k},k}\right)\right)q_{y_k}\left(t_{M_{1,k},k}\right),\end{aligned}\right\} \quad (18)$$

$$\left.\begin{aligned}&Du_k^0(t_{1,k}) = \Pi(t_{1,k},\sigma_k(t_{1,k}))q_{x_k y_k}(t_{1,k}),\\&D\left(u_k^0(t_{1,k})+u_k^0(t_{2,k})\right) = \Pi(t_{2,k},\sigma_k(t_{2,k}))q_{x_k y_k}(t_{2,k}),\\&\dots\dots\dots\dots\dots\dots,\\&D\sum_{m=1}^{M_{1,k}} u_k^0(t_{m,k}) = \Pi\left(t_{M_{1,k},k},\sigma_k\left(t_{M_{1,k},k}\right)\right)q_{x_k y_k}\left(t_{M_{1,k},k}\right),\end{aligned}\right\} \quad (19)$$

where $D = -\frac{1+\kappa}{2\mu}\frac{\pi l_k}{M}$, $k=1,2,\dots,N$.

The obtained systems (16), (18), (19) turned to be associated and should be solved jointly. As the stresses in the disk are restricted, the solutions of integral equations are sought in the class of everywhere bounded functions. Such a solution exists subject to solvability conditions of integral equations.

For the closeness of the obtained algebraic equations, we miss $2\times N$ equations expressing the solvability conditions of integral equations (stress finiteness conditions in the vicinity of the tips of the cracks with end zones). Writing these conditions, we get one more $2\times N$ complex equations

$$\left.\begin{aligned}&\sum_{m=1}^{M}(-1)^m g_n^0(t_m)\cot\frac{2m-1}{4M}\pi = 0,\\&\sum_{m=1}^{M}(-1)^{M+m} g_n^0(t_m)\tan\frac{2m-1}{4M}\pi = 0,\\&(n=1,2,\dots,N).\end{aligned}\right\} \quad (20)$$

The obtained relations (16), (18), (19), (20) permit to get the terminal solution of the problem if the coefficients α_k^* $(k=0,\pm1,\dots)$ are determined. For composing an infinite system of linear algebraic equations with respect to the unknowns α_k^*, subject to (14) we substitute (13) in condition (7). After some transformations, condition (7) is reduced to the form

$$\sum_{m=0}^{\infty}A_m\left(\frac{\tau}{R}\right)^m + \sum_{m=0}^{\infty}A_m^*\left(\frac{R}{\tau}\right)^m = \\ = \sum_{m=0}^{\infty}U_m\left(\frac{\tau}{R}\right)^m + \sum_{m=0}^{\infty}U_m^*\left(\frac{R}{\tau}\right)^m. \quad (21)$$

Because of some awkwardness of the expressions for A_m, A_m^*, U_m, U_m^* $(m=0,1,2,\dots)$, they are not cited.

Comparing in the both sides of the obtained relation (21) the coefficient, with the identical powers τ/R and R/τ, we find two infinite systems of linear algebraic equations:

$$\left.\begin{aligned}&A_0 + A_0^* = U_0 + U_0^* \quad (m=0),\\&A_m = U_m \quad (m=1,2,\dots),\\&A_m^* = U_m^* \quad (m=1,2,\dots).\end{aligned}\right\} \quad (22)$$

The joint solution of the obtained equations permits under the given characteristics of bonds to determine the forces in bonds, the sizes of the end zones and also the stress-strain state of the disk in the presence of arbitrary number of cracks with end zones.

For formulation of the limit equilibrium criterion we use the criterion of critical opening of the crack surfaces. We can determine the opening of the crack faces within the end zones by the relations

$$\begin{aligned}&v_k^+(x_k,0) - v_k^-(x_k,0) = \Pi(x_k,\sigma_k)q_{y_k}(x_k) \quad \text{on } L_k'',\\&u_k^+(x_k,0) - u_k^-(x_k,0) = \Pi(x_k,\sigma_k)q_{x_k y_k}(x_k)\\&k=1,2,\dots,N.\end{aligned}$$

The condition of critical opening of the crack faces at the edge of the end zone will be

$$\left.\begin{aligned}&\Pi(l_{0k},\sigma(l_{0k}))\sigma(l_{0k}) = \delta_c, \qquad x_k = l_{0k},\\&\Pi(-l_{0k},\sigma(-l_{0k}))\sigma(-l_{0k}) = \delta_c \qquad x_k = -l_{0k},\end{aligned}\right\} \quad (23)$$

where δ_c is the fracture toughness of the disk material to be determined experimentally.

4. Method of numerical solution and analysis

For numerical realization of the obtained solution it is necessary the joint solution of equations (16), (18), (19), (20), (22) and (23). Because of unknown sizes of the prefracture end zones even at linearly elastic bonds the systems of algebraic equations became nonlinear. In this connection, for solving the obtained systems in the case of linear bonds, the successive approximations method was used.

We solve the combined system for some certain values of l_k^* $(k=1,2,\ldots,N)$ with respect to the unknowns α_k^*, $g_k^0(t_m)$, $q_{y_k}(t_{m,k})$ and $q_{x_k y_k}(t_{m,k})$. The values of l_k^* and the found quantities α_k^*, $g_k^0(t_m)$, $q_{y_k}(t_{m,k})$ and $q_{x_k y_k}(t_{m,k})$ are substituted in (20), i.e. into the unused equations of the combined system. The taken values of the parameters l_k^* and the corresponding values of α_k^*, $g_k^0(t_m)$, $q_{y_k}(t_{m,k})$, $q_{x_k y_k}(t_{m,k})$ will not, generally speaking, satisfy the equations (20). Therefore, by choosing the values of the parameters l_k^* we will repeat the calculations over and over again until equations (20) of the combined system will be satisfied with the given accuracy. The combined system of equations at each approximation was solved by the Gauss method with the choice of the principal element for various values of *M*.

In the case of nonlinear law of deformation of bonds, for finding the forces at the end zones, an algorithm similar to the A.A. Il'yushin method of elastic solutions [13] was used. The effective compliance calculation is conducted as in definition of the secant modulus in the method of variable elasticity parameters. The successive approximations process ends as soon as the forces along the end zone, obtained at two successive iterations differ little from each other.

The nonlinear part of the curve of deformation of bonds was taken in the form of bilinear dependence whose ascending part corresponded to elastic deformation of bonds $(0<V(x_k)<V_*)$ with maximum tension of bonds. Here $V(x_k)=\left|(u_k^+-u_k^-)-i(v_k^+-v_k^-)\right|$. For $V(x_k)>V_*$ the law of deformation was described by a nonlinear dependence determined by the two points (V_*,σ_*) and (δ_c,σ_c), moreover for $\sigma_c \ge \sigma_*$ we have an ascending linear dependence (linear hardening corresponding to elasto-plastic deformation of bonds).

The graphs of dependence of dimensionless length of the end zone for the left end of the crack from the dimensionless parameter N_0/σ_* for the following values of free values of free parameters $\varepsilon=l_{01}/R=0.05$; 0.10; $\alpha_1=\pi/4$; $z_1^0=0.1Re^{i\pi/18}$; N_0 is a force factor, are depicted in Fig. 2.

The graphs of distribution of normal forces q_{y_1}/N_0 in the bonds for the right crack tip zone (curve *1* for linear deformations of bonds, curve *2* for a bilinear curve of deformation of bonds) are given in Fig. 3.

The calculations show that at linear law of deformation of bonds, the forces in bonds have always maximum values at the edge of the end zone. The similar picture is observed for the values of opening of the crack faces as well. Therewith, with increase of relative compliance and for the values of opening of the crack faces. Therewith, the opening of the crack increases according to increase of relative compliance of bonds.

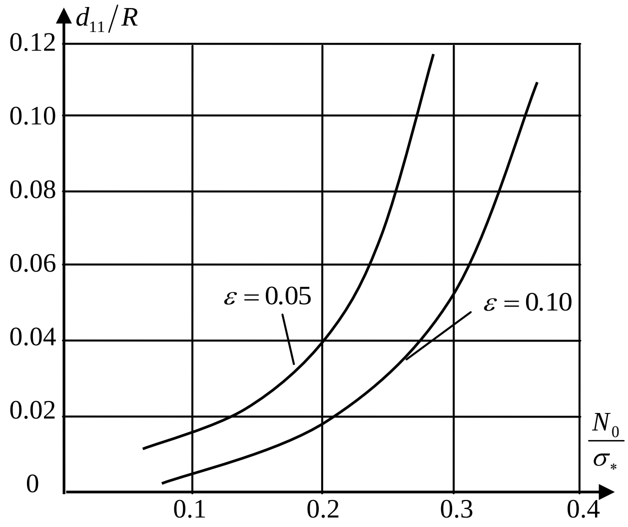

Fig. 2 Dependence of dimensionless length of the end zone for the left end of the crack from the dimensionless parameter N_0/σ_*

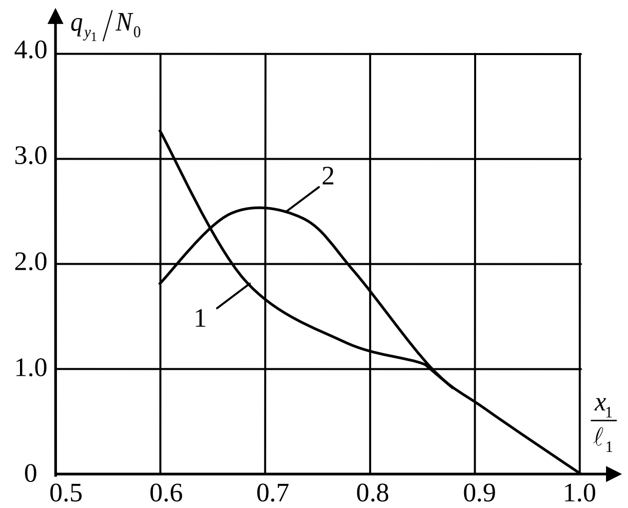

Fig. 3 Distribution of normal forces q_{y_1}/N_0 in the bonds for the right crack tip zone

5. Conclusions

The obtained closed algebraic system of equations and limit condition of the crack growth permits by means of numerical calculation and for each specific circular disk to set up admissible size of cracks for different laws of deformation of interparticle bonds, elastic and geometrical characteristics of the material and disk. The developed calculation method permits to solve the following practically important problems:

1) to estimate the guaranteed life of a circular disk with regard to expected defects and loading conditions;

2) to set up admissible level of defects and maximum value of workloads providing sufficient safety margin;

3) to conduct the choice of a disk material with a complex characteristics of fracture toughness.

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INTERACTION OF BRIDGED CRACKS IN A CIRCULAR DISK WITH MIXED BOUNDARY CONDITIONS

S u m m a r y

Interaction of arbitrarily located system of rectilinear bridged cracks is considered. It is assumed that the mixed boundary conditions are given on the boundary of the circular disk. It is accepted that the fracture zone is a finite length layer containing a material with partially disturbed bonds between its separate structural elements (end zone). Existence of bonds between the cracks surfaces in the end zones is simulated by application of cohesive forces caused by the presence of bonds to the crack surfaces. Limit equilibrium analysis of cracks is formulated, taking account the criterion of the limit traction of the bonds in the material at the edge of the cracks end zone.

Keywords: circular disk, mixed boundary conditions, interaction of the system of cracks, cracks with interfacial bonds, cohesive forces.

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